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Worldsheet instantons and torsion curves. Part B: mirror symmetry

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Abstract: We apply mirror symmetry to the problem of counting holomorphic rational curves in a Calabi-Yau threefold X with $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ Wilson lines. As we found in Part A [1], the integral homology group $H_2(X, \mathbb{Z}) = \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ contains *torsion* curves. Using the B-model on the mirror of X as well as its covering spaces, we compute the instanton numbers. We observe that X is self-mirror even at the quantum level. Using the self-mirror property, we derive the complete prepotential on X, going beyond the results of Part A. In particular, this yields the first example where the instanton number depends on the torsion part of its homology class. Another consequence is that the threefold X provides a non-toric example for the conjectured exchange of torsion subgroups in mirror manifolds.

Keywords: Topological Strings, Superstrings and Heterotic Strings, Solitons Monopoles and Instantons.

JHEP10(2007)023

Contents

1. Introduction

Counting world sheet instantons (that is, holomorphic curves) on a Calabi-Yau threefold has had a large number of applications in mathematics and physics, ever since it was essentially solved by mirror symmetry several years ago [2]. The purpose of this paper is to take into account an important subtlety that does not appear in very simple Calabi-Yau manifolds like hypersurfaces in smooth toric varieties. This subtlety is the appearance of *torsion* curve classes. That is, the homology[1] group

$$H_2\big(X, \mathbb{Z}\big) = \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \tag{1.1}$$

contains the *torsion*[2] *subgroup* $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. Here, the manifold of interest X is a quotient of one of Schoen's Calabi-Yau manifolds [3, 4] by a freely acting symmetry group. There are already a few known examples of such Calabi-Yau manifolds with torsion curves [5 − 9], but the proper instanton counting has never been done before.

The prime motivation for studying these curves is that one would like to compute the superpotential for the vector bundle moduli [10 − 16] in a heterotic MSSM [17 − 25]. Our main result will be that there exist smooth rigid rational curves in X that are alone in their homology class. This proves that, in general, no cancellation between contributions to the superpotential W from instantons in the same homology class can occur.

Therefore we would like to count rational curves on X. In physical terms, we need to find the instanton correction $\mathcal{F}^{\mathrm{np}}_{X,0}$ to the genus zero prepotential of the (A-model) topological string on X. This is usually written as a (convergent) power series in h^{11} variables $q_a = e^{2\pi \mathrm{i} t^a}$. Each summand is the contribution of an instanton, and the (integer) coefficients are the multiplicities of instantons in each homology class. According to [26, 27, 1] the novel feature of the 3-torsion curves on X is that for each 3-torsion generator we need an additional variable b_j such that $b_j^3 = 1$. The Fourier series of the prepotential on X becomes

$$\mathcal{F}^{\mathrm{np}}_{X,0}(p, q, r,\ b_1, b_2) = \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z} \\ m_1,m_2 \in \mathbb{Z}_3}} n_{(n_1,n_2,n_3,m_1,m_2)} \operatorname{Li}_3\left(p^{n_1} q^{n_2} r^{n_3} b_1^{m_1} b_2^{m_2}\right), \tag{1.2}$$

where $n_{(n_1,n_2,n_3,m_1,m_2)}$ is the instanton number in the curve class $(n_1, n_2, n_3, m_1, m_2)$.

For the purpose of computing the prepotential, we can either use directly the A-model or start with the B-model and apply mirror symmetry. The A-model calculation was carried out in the companion paper [1], entitled Part A. The results were:

- A set of powerful techniques to compute the torsion subgroups in the integral homology and cohomology groups of X. They are spectral sequences starting with the so-called group (co)homology of the group action on the universal cover $\widetilde{X}$.

[1] In the following, $\mathbb{Z}_3 \stackrel{\text{def}}{=} \mathbb{Z}/3\mathbb{Z}$ always denotes the integers mod 3. Similarly, we write $(\mathbb{Z}_3)^n = \oplus_n \mathbb{Z}_3 = \mathbb{Z}_3 \oplus \cdots \oplus \mathbb{Z}_3$ for the Abelian group generated by n generators of order 3.

[2] Not to be confused with the torsion tensor of a connection.

JHEP10(2007)023

- A closed formula for the genus zero prepotential

$$\mathcal{F}^{\mathrm{np}}_{X,0}(p,q,r,b_1,b_2) = \left(\sum_{i,j=0}^{2} p b_1^i b_2^j\right) P(q)^4 P(r)^4 + O(p^2) = \sum_{i,j=0}^{2} \mathrm{Li}_3(p b_1^i b_2^j) + \cdots \quad (1.3)$$

  to linear order in p, extending the one computed in [28] for the universal cover $\widetilde{X}$. Here, if $p(k)$ is the number of partitions of $k \in \mathbb{Z}_{\geq}$, then $P(q)$ is the generating function for partitions,

$$P(q) \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} p(i) q^i = \frac{q^{\frac{1}{24}}}{\eta(\frac{1}{2\pi \mathrm{i}} \ln q)}. \quad (1.4)$$

- Expanding eq. (1.3) as an instanton series we find that the number of rational curves of degree $(1,0,0,m_1,m_2)$ is:

$$n_{(1,0,0,m_1,m_2)} = 1, \qquad \forall\, m_1, m_2 \in \mathbb{Z}_3. \quad (1.5)$$

  Furthermore, these curves have normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Hence, there are indeed 9 smooth rigid rational curves which are alone in their homology class.

Alternatively, one can start with the B-model topological string and apply mirror symmetry, which is what we will do in this paper, entitled Part B. This will allow us to obtain the higher order terms in p. The order in p up to which one wants to compute the instanton numbers is only limited by computer power. We will again find a closed formula at every order in p, however, this time by guessing it from the instanton calculation, and hence only up to the order given by this limitation. The way to arrive at this result is as follows:

- The universal cover $\widetilde{X}$ admits a simple realization as a complete intersection in a toric variety. In this situation, mirror symmetry boils down to an algorithm to compute instanton numbers. Unfortunately, there are many non-toric divisors which cannot be treated this way. It turns out that, after descending to X, precisely the torsion information is lost. In this approach one can only compute $\mathcal{F}^{\mathrm{np}}_{X,0}(q_1,q_2,q_3,\,1,1)$.

- As a pleasant surprise we find strong evidence that the manifold X is self-mirror. In particular, we attempt to compute the instanton numbers on the mirror X^* by descending from the covering space $\widetilde{X}^*$. The embedding of $\widetilde{X}^*$ into a toric variety is such that all 19 divisors are toric. In principle, this allows for a complete analysis including the full $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ torsion information, but this is too demanding in view of current computer power.

- Although the full quotient $X = \widetilde{X}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is not toric, it turns out that a certain partial quotient $\overline{X} = \widetilde{X}/\mathbb{Z}_3$ can be realized as a complete intersection in a toric variety. That way, one only has to deal with $h^{11}(\overline{X}) = 7$ parameters, which is manageable on a desktop computer. Assuming the self-mirror property, we work with the mirror $\overline{X}^*$, for which again all divisors are toric, and we can compute the expansion of $\mathcal{F}^{\mathrm{np}}_{X,0}(p,q,r,\,1,b_2)$ to any desired degree. A symmetry argument then allows one to recover the b_1 dependence as well. Finally, we can extract the instanton numbers $n_{(n_1,n_2,n_3,m_1,m_2)}$ including the torsion information.

- As can be seen from the A-model result eq. (1.3), we observe that the prepotential $\mathcal{F}^{\rm np}_{X,0}$ at order p factors into $\sum_{i,j=0}^{2} b_1^i b_2^j$ times a function of p, q, r only. This means that the instanton number does not depend on the torsion part of its homology class. We will explain the underlying reason for this factorization and show that it breaks down at order p^3. This fits nicely with the B-model computation at order p^3, where the instanton numbers *do* depend on the torsion part.

- Another consequence of the self-mirror property is that X is a non-toric example for the conjecture of [6]. By this conjecture, certain torsion subgroups of the integral homology groups are exchanged under mirror symmetry.

An easily readable overview and a discussion of the physical consequences of our findings for superpotentials and moduli stabilization of heterotic models was presented in [27]. The present Part B is self-contained and can be read independently of Part A [1]. All necessary results from Part A are reproduced in this part.

As a guide through this paper, we start in section 2 with a brief overview of the topology of the various spaces involved as determined in Part A [1]. This is followed by a review of the Batyrev-Borisov construction of mirror pairs of complete intersections in toric varieties in section 3. We illustrate this construction by means of the covering spaces $\widetilde{X}$ and $\widetilde{X}^*$ of our example. The review includes the techniques to compute the B-model prepotential and the mirror map. These are applied in section 4 to the partial quotients $\overline{X}$ and $\overline{X}^*$ yielding the main results stated above. This assumes that X as well as $\overline{X}$ are self-mirror, and evidence for this property is recapitulated in section 5. Moreover, we show how the torsion subgroups are exchanged. Section 6 contains an explanation for the breakdown of the factorization alluded to above. Putting all the information together we try to guess a closed form for the prepotential in section 7. Finally, we present our conclusions in section 8. In the course of this work we will notice that a certain flop of X is very natural from the toric point of view, and we will present it in appendix B.

2. Calabi-Yau threefolds

2.1 The Calabi-Yau threefold X

The Calabi-Yau manifold X of interest is constructed as a free $G \stackrel{\rm def}{=} \mathbb{Z}_3 \times \mathbb{Z}_3$ quotient of its universal covering space $\widetilde{X}$. The latter is one of Schoen's Calabi-Yau threefolds [3]. It is simply connected and hence easier to study. Among its various descriptions are the fiber product of two dP_9 surfaces, a resolution of a certain T^6 orbifold [29], or a complete intersection in a toric variety. In the present Part B, we will mostly use the latter viewpoint. The simplest way is to introduce the toric ambient variety $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$ with homogeneous coordinates

$$\Big([x_0 : x_1 : x_2],\ [t_0 : t_1],\ [y_0 : y_1 : y_2]\Big) \in \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2\,. \tag{2.1}$$

The embedded Calabi-Yau threefold $\widetilde{X}$ is then obtained as the complete intersection of a degree $(0,1,3)$ and a degree $(3,1,0)$ hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$. We restrict the coefficients of their defining equations $F_i = 0$ to a particular set of three complex parameters

λ_1, λ_2, λ_3, such that the polynomials F_i read

$$t_0\Big(x_0^3 + x_1^3 + x_2^3\Big) + t_1\Big(x_0 x_1 x_2\Big) \stackrel{\text{def}}{=} F_1 \tag{2.2a}$$

$$\big(\lambda_1 t_0 + t_1\big)\Big(y_0^3 + y_1^3 + y_2^3\Big) + \big(\lambda_2 t_0 + \lambda_3 t_1\big)\Big(y_0 y_1 y_2\Big) \stackrel{\text{def}}{=} F_2. \tag{2.2b}$$

For the special complex structure parametrized by λ_1, λ_2, λ_3 the complete intersection is invariant under the $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ action generated by ($\zeta \stackrel{\text{def}}{=} e^{\frac{2\pi i}{3}}$)

$$g_1 : \begin{cases} [x_0 : x_1 : x_2] \mapsto [x_0 : \zeta x_1 : \zeta^2 x_2] \\ [t_0 : t_1] \mapsto [t_0 : t_1] \text{ (no action)} \\ [y_0 : y_1 : y_2] \mapsto [y_0 : \zeta y_1 : \zeta^2 y_2] \end{cases} \tag{2.3a}$$

and

$$g_2 : \begin{cases} [x_0 : x_1 : x_2] \mapsto [x_1 : x_2 : x_0] \\ [t_0 : t_1] \mapsto [t_0 : t_1] \text{ (no action)} \\ [y_0 : y_1 : y_2] \mapsto [y_1 : y_2 : y_0] \end{cases} \tag{2.3b}$$

One can show that the fixed points of this group action in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$ do not satisfy eqs. (2.2a) and (2.2b), hence the action on $\widetilde{X}$ is free.

2.2 The intermediate quotient $\overline{\mathbf{X}}$

The partial quotient

$$\overline{X} \stackrel{\text{def}}{=} \widetilde{X}/G_1 \tag{2.4}$$

will be of particular interest in this paper because this quotient is generated by phase symmetries, see eq. (2.3a), and hence is toric. In particular, we will need a basis of Kähler classes. As usual, we will not distinguish degree-2 cohomology and degree-4 homology classes but identify them via Poincaré duality. Part A [1] subsection 5.3 shows that[3]

$$H^2\big(\overline{X}, \mathbb{Z}\big) = H^2(\widetilde{X}, \mathbb{Z})^{G_1} \oplus \mathbb{Z}_3 = \operatorname{span}_{\mathbb{Z}}\Big\{\phi,\ \tau_1,\ \upsilon_1,\ \psi_1,\ \tau_2,\ \upsilon_2,\ \psi_2\Big\} \oplus \mathbb{Z}_3. \tag{2.5}$$

Hence, by abuse of notation, we can identify the free generators on $\overline{X}$ with the G_1-invariant generators on $\widetilde{X}$, see Part A eq. (8.48), via the pull back by the quotient map. The triple intersection numbers on $\overline{X} = \widetilde{X}/\mathbb{Z}_3$ are one-third of the corresponding intersection numbers

[3] The torsion in H^2 are just the Wilson lines, that is, first Chern classes of flat line bundles. They will play no role in the following. The torsion curves in H_2, on the other hand, are the focus of this paper.

JHEP10(2007)023

Calabi-Yau threefold	$H_2(-,\mathbb{Z})$	Free generators	Torsion generators
$\widetilde{X}$	$\mathbb{Z}^{19}$	$\{p_0, q_0, \dots, q_8, r_0, \dots, r_8\}$	$\varnothing$
$\overline{X} = \widetilde{X}/G_1$	$\mathbb{Z}^7 \oplus \mathbb{Z}_3$	$\{P, Q_1, Q_2, Q_3, R_1, R_2, R_3\}$	$\{b_1\}$
$X = \widetilde{X}/G$	$\mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$	$\{p, q, r\}$	$\{b_1, b_2\}$

Table 1: The different Calabi-Yau threefolds, curve classes, and variables used to expand the prepotential.

on $\widetilde{X}$ listed in Part A eq. (8.50). Hence, the intersection numbers on $\overline{X}$ are

$$\begin{array}{rrrrr}
\phi\tau_1\tau_2 = 3 & \phi\tau_1\upsilon_2 = 3 & \phi\tau_1\psi_2 = 6 & \phi\upsilon_1\tau_2 = 3 & \phi\upsilon_1\upsilon_2 = 3 \\
\phi\upsilon_1\psi_2 = 6 & \phi\psi_1\tau_2 = 6 & \phi\psi_1\upsilon_2 = 6 & \phi\psi_1\psi_2 = 12 & \tau_1^2\tau_2 = 1 \\
\tau_1^2\upsilon_2 = 1 & \tau_1^2\psi_2 = 2 & \tau_1\upsilon_1\tau_2 = 3 & \tau_1\upsilon_1\upsilon_2 = 3 & \tau_1\upsilon_1\psi_2 = 6 \\
\tau_1\psi_1\tau_2 = 3 & \tau_1\psi_1\upsilon_2 = 3 & \tau_1\psi_1\psi_2 = 6 & \tau_1\tau_2^2 = 1 & \tau_1\tau_2\upsilon_2 = 3 \\
\tau_1\tau_2\psi_2 = 3 & \tau_1\upsilon_2^2 = 3 & \tau_1\upsilon_2\psi_2 = 6 & \tau_1\psi_2^2 = 6 & \upsilon_1^2\tau_2 = 3 \\
\upsilon_1^2\upsilon_2 = 3 & \upsilon_1^2\psi_2 = 6 & \upsilon_1\psi_1\tau_2 = 6 & \upsilon_1\psi_1\upsilon_2 = 6 & \upsilon_1\psi_1\psi_2 = 12 \\
\upsilon_1\tau_2^2 = 1 & \upsilon_1\tau_2\upsilon_2 = 3 & \upsilon_1\tau_2\psi_2 = 3 & \upsilon_1\upsilon_2^2 = 3 & \upsilon_1\upsilon_2\psi_2 = 6 \\
\upsilon_1\psi_2^2 = 6 & \psi_1^2\tau_2 = 6 & \psi_1^2\upsilon_2 = 6 & \psi_1^2\psi_2 = 12 & \psi_1\tau_2^2 = 2 \\
\psi_1\tau_2\upsilon_2 = 6 & \psi_1\tau_2\psi_2 = 6 & \psi_1\upsilon_2^2 = 6 & \psi_1\upsilon_2\psi_2 = 12 & \psi_1\psi_2^2 = 12.
\end{array} \tag{2.6}$$

Clearly, G_2 acts on the partial quotient $\overline{X}$. From Part A eq. (8.54) it follows that, of the 7 non-torsion divisors above, only 3 are G_2-invariant. This invariant part is particularly manageable and will be important in the following. We find

$$H^2\big(\widetilde{X},\mathbb{Z}\big)^G_{\text{free}} = H^2\big(\overline{X},\mathbb{Z}\big)^{G_2}_{\text{free}} = \operatorname{span}_{\mathbb{Z}}\Big\{\phi,\tau_1,\tau_2\Big\} \tag{2.7}$$

with products $3\tau_i^2 = \tau_i\phi$. In particular, the triple intersection numbers on $\overline{X}$ are

$$\tau_1^2\tau_2 = 1, \quad \tau_1\phi\tau_2 = 3, \quad \tau_1\tau_2^2 = 1, \tag{2.8}$$

and 0 otherwise. Finally, the second Chern class of $\overline{X}$ is $c_2(\overline{X}) = 12(\tau_1^2 + \tau_2^2)$. Therefore,

$$c_2\big(\overline{X}\big)\cdot\tau_1 = 12, \quad c_2\big(\overline{X}\big)\cdot\phi = 0, \quad c_2\big(\overline{X}\big)\cdot\tau_2 = 12. \tag{2.9}$$

2.3 Variables

As we discussed in Part A subsection 8.1, the instanton-generated superpotential should be thought of as a series with one variable for each generator in H_2.

In particular, we will be interested in the Calabi-Yau threefolds $\widetilde{X}$, $\overline{X}$, and X. For these, the degree-2 integral homology and the variables used (see Part A [1] for precise definitions) are in summarized table 1. Pushing down the curves by the respective quotients

JHEP10(2007)023

lets us express the prepotential on the quotient in terms of the prepotential on the covering space. We found in Part A that

$$\begin{aligned}\mathcal{F}^{\text{np}}_{\overline{X},0}(P,Q_1,Q_2,Q_3,R_1,R_2,R_3,b_1) =& \frac{1}{|G_1|}\mathcal{F}^{\text{np}}_{\widetilde{X},0}\Big(PQ_1^5Q_2^6R_1^5R_2^6,\\ & Q_1^5Q_2^6,\, Q_1^{-2}Q_2^{-2}Q_3^{-3}b_1,\, Q_1^{-1}Q_2^{-1},\, Q_3^3,\, Q_3^2b_1,\, Q_3,\, 1,\, b_1,\, Q_1Q_3^3,\\ & R_1^5R_2^6,\, R_1^{-2}R_2^{-2}R_3^{-3}b_1^2,\, R_1^{-1}R_2^{-1},\, R_3^3,\, R_3^2b_1^2,\, R_3,\, 1,\, b_1^2,\, R_1R_3^3\Big)\end{aligned} \tag{2.10}$$

and

$$\mathcal{F}^{\text{np}}_{X,0}(p,q,r,b_1,b_2) = \frac{1}{|G_2|}\mathcal{F}^{\text{np}}_{\overline{X},0}(p,\, q,\, b_2,\, b_2,\, r,\, b_2^2,\, b_2^2,\, b_1). \tag{2.11}$$

3. Toric geometry and mirror symmetry

In this section we review mirror symmetry and the construction of the B-model for the mirror of the covering space $\widetilde{X}$. Since $\widetilde{X}$ is a complete intersection in a toric variety, we can use the standard constructions. Because we expect the model to be self-mirror, we will analyze the B-model for $\widetilde{X}$ and its mirror $\widetilde{X}^*$. The toric geometry for $\widetilde{X}$ is much simpler[4] than for $\widetilde{X}^*$, but contains less information. In this section we will start with the simpler model in order to review the Batyrev-Borisov construction for the mirror of a complete intersection in a toric variety. Then we will apply this construction to the more complicated model, now without going into too many details. We will see that, on the simpler side, not all parameters are toric and no torsion is visible. However, on the more complicated mirror side, all parameters are toric which will allow us, in principle, to perform the B-model computation of the complete prepotential. As $\widetilde{X} \cong \widetilde{X}^*$ is expected to be self-mirror, this determines the complete prepotential $\mathcal{F}^{\text{np}}_{\widetilde{X},0} = \mathcal{F}^{\text{np}}_{\widetilde{X}^*,0}$ as well. In practice, however, the analysis is computationally too involved.

Fortunately, the space $\overline{X} = \widetilde{X}/G_1$ and its mirror will turn out to be both tractable with toric methods and sufficiently informative. This quotient will be the subject of section 4. Finally, this is also the starting point for arguing in section 5 that the self-mirror property persists at the level of instanton corrections.

Recall that, in subsection 2.1 we defined our Calabi-Yau manifold as the complete intersection

$$\widetilde{X} \stackrel{\text{def}}{=} \Big\{F_1 = 0,\, F_2 = 0\Big\} \;\subset\; \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2 \tag{3.1}$$

with the two polynomials F_1, F_2 as in eqs. (2.2a) and (2.2b), respectively. In order to construct the mirror manifold following Batyrev and Borisov, we need to reformulate this definition in terms of toric geometry. We review here some essential ingredients of toric geometry, for details we refer to [30, 31] and references therein. We will give the abstract definitions and concepts step by step, and at each step illustrate them with the example $\widetilde{X}$ and its mirror manifold $\widetilde{X}^*$.

[4] Meaning that $\widetilde{X}$ is a complete intersection in the very simple toric variety $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$, whereas $\widetilde{X}^*$ is embedded in a complicated toric ambient variety.

3.1 Toric varieties

Given a lattice N of dimension d, a toric variety V_Σ is defined in terms of a fan Σ which is a collection of rational polyhedral[5] cones $\sigma \subset N$ such that it contains all faces and intersections of its elements. V_Σ is compact if the support of Σ covers all of the real extension $N_\mathbb{R}$ of the lattice N. The resulting d-dimensional variety V_Σ is smooth if all cones are simplicial and if all maximal cones are generated by a lattice basis.

Let $\Sigma^{(1)}$ denote the set of one-dimensional cones (rays) with primitive generators ρ_i, $i = 1, \dots, n$. The simplest description of V_Σ introduces n homogeneous coordinates z_i corresponding to the generators ρ_i of the rays in $\Sigma^{(1)}$. These homogeneous coordinates are then subjected to weighted projective identifications

$$\big[z_1 : \cdots : z_n\big] = \big[\lambda^{q_1^{(a)}} z_1 : \cdots : \lambda^{q_n^{(a)}} z_n\big] \qquad a = 1, \dots, h \tag{3.2}$$

for any nonzero complex number $\lambda \in \mathbb{C}^\times$, where the integer n-vectors $q_i^{(a)}$ are generators of the linear relations $\sum q_i^{(a)} \rho_i = 0$ among the primitive lattice vectors[6] ρ_i. In order to obtain a well-behaved quotient, we must exclude an exceptional set $Z(\Sigma) \subset \mathbb{C}^n$ that is defined in terms of the fan, as will be explained below. Hence, the quotient is

$$V_\Sigma = \Big(\mathbb{C}^n - Z(\Sigma)\Big)\Big/\Big(\big(\mathbb{C}^\times\big)^h \times \Gamma\Big), \tag{3.3}$$

where $\Gamma \simeq N/\operatorname{span}\{\rho_i\}$ is a finite abelian group. There are $h = n - d$ independent $\mathbb{C}^\times$ identifications, therefore the complex dimension of V_Σ equals the rank d of the lattice N. The identifications by Γ are only non-trivial if the ρ_i do not span the lattice N. Refinements of the lattice N with fixed ρ_i can hence be used to construct quotients of toric varieties V_Σ by discrete phase symmetries such as $\mathbb{Z}_3$. Such quotients will be discussed in section 4. Note that the rays ρ_i are in 1-to-1 correspondence with the $(\mathbb{C}^\times)$-invariant divisors D_i on V_Σ, which are defined as

$$D_i = \big\{z_i = 0\big\} \ \subset V_\Sigma. \tag{3.4}$$

Conversely, the homogeneous coordinate z_i is a section of the line bundle $\mathcal{O}(D_i)$.

For example, consider the simplest compact toric variety, the projective space $\mathbb{P}^d$. Its fan $\Sigma = \Sigma(\Delta)$ is generated by the $n = d + 1$ vectors

$$\rho_1 = e_1, \quad \rho_2 = e_2, \quad \dots, \quad \rho_{n-1} = e_d, \quad \rho_n = -\sum_{i=1}^{d} e_i \tag{3.5}$$

of a d-dimensional simplex Δ. They satisfy a single linear relation, $\sum_{i=1}^n \rho_i = 0$. Therefore $q_i = 1$ for all i, and the homogeneous coordinates in eq. (3.2) are the usual homogeneous coordinates on $\mathbb{P}^d$.

[5] Here, the tip of the cone is always the origin of N. A cone is rational if it is spanned by rays which pass through lattice points (other than the origin), that is, have rational slopes. A cone is polyhedral if it is the cone over an $(d-1)$-dimensional polytope. In other words, curved faces are not allowed.

[6] We will use the same symbol ρ_* to denote the generators in $\Sigma^{(1)}$ and the corresponding primitive lattice vectors in N.

JHEP10(2007)023

For products of toric varieties we simply extend the relations for any single factor by zeros and take the union of them. Hence, the fan of the polyhedron Δ^* describing the 5-dimensional toric variety $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$ in eq. (3.1) is generated by the $n = 5 + 3 = 8$ vectors

$$\begin{aligned} &\rho_1 = e_1, \quad \rho_2 = e_2, \quad \rho_3 = -e_1 - e_2, \quad \rho_4 = e_3, \\ &\rho_5 = -e_3, \quad \rho_6 = e_4, \quad \rho_7 = e_5, \quad \rho_8 = -e_4 - e_5 \end{aligned} \tag{3.6}$$

satisfying the linear relations

$$\sum_{i=1}^{3} \rho_i = \sum_{i=4}^{5} \rho_i = \sum_{i=6}^{8} \rho_i = 0. \tag{3.7}$$

Except for the origin, there are no other lattice points in the interior of Δ^*. The corresponding homogeneous coordinates will be denoted by

$$\begin{aligned} &z_1 = x_0, && z_2 = x_1, && z_3 = x_2, \\ &z_4 = t_0, && z_5 = t_1, && \\ &z_6 = y_0, && z_7 = y_1, && z_8 = y_2. \end{aligned} \tag{3.8}$$

In more general situations, given a polytope $\Delta^* \subset N$ we will denote the resulting toric variety by $\mathbb{P}_{\Delta^*} = V_{\Sigma(\Delta^*)}$.

3.2 The Batyrev-Borisov construction

Batyrev showed that a generic section of $K^{-1}_{\mathbb{P}_{\Delta^*}}$, the anticanonical bundle of $\mathbb{P}_{\Delta^*}$, defines a Calabi-Yau hypersurface if Δ^* is reflexive, which means, by definition, that Δ^* and its dual

$$\Delta = \left\{ x \in M_{\mathbb{R}} \,\middle|\, (x, y) \geq -1 \ \forall y \in \Delta^* \right\} \tag{3.9}$$

are both lattice polytopes. Here, $M = \mathrm{Hom}(N, \mathbb{Z})$ is the lattice dual to N and $M_{\mathbb{R}}$ is its real extension. Mirror symmetry corresponds to the exchange of Δ and Δ^* [32]. The generalization of this construction to complete intersections of codimension $r > 1$ is due to Batyrev and Borisov [33, 34]. For that purpose, they introduced the notion of a *nef partition*. Consider a dual pair of d-dimensional reflexive polytopes $\Delta \subset M_{\mathbb{R}}, \Delta^* \subset N_{\mathbb{R}}$. In that context, a partition $E = E_1 \cup \cdots \cup E_r$ of the set of vertices of Δ^* into disjoint subsets $E_1, \ldots, E_r$ is called a nef-partition if there exist r integral upper convex $\Sigma(\Delta^*)$-piecewise linear support functions $\phi_l : N_{\mathbb{R}} \to \mathbb{R}$, $l = 1, \ldots, r$ such that

$$\phi_l(\rho) = \begin{cases} 1 & \text{if } \rho \in E_l, \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Each ϕ_l corresponds to a divisor

$$D_{0,l} = \sum_{\rho \in E_l} D_\rho \tag{3.11}$$

on $\mathbb{P}_{\Delta^*}$, and their intersection

$$Y = D_{0,1} \cap \cdots \cap D_{0,r} \tag{3.12}$$

defines a family Y of Calabi-Yau complete intersections of codimension r. Moreover, each ϕ_l corresponds to a lattice polyhedron Δ_l defined as

$$\Delta_l = \left\{ x \in M_{\mathbb{R}} \,\middle|\, (x,y) \geq -\phi_l(y) \quad \forall y \in N_{\mathbb{R}} \right\}. \tag{3.13}$$

The lattice points $m \in \Delta_l$ correspond to monomials

$$z^m = \prod_{i=1}^{n} z_i^{\langle m,\rho_i \rangle} \quad \in \Gamma\left(\mathbb{P}_{\Delta^*}, \mathcal{O}(D_{0,l})\right). \tag{3.14}$$

One can show that the sum of the functions ϕ_l is equal to the support function of $K^{-1}_{\mathbb{P}_{\Delta^*}}$ and, therefore, the corresponding Minkowski sum is $\Delta_1 + \cdots + \Delta_r = \Delta$. Moreover, the knowledge of the decomposition $E = E_1 \cup \cdots \cup E_r$ is equivalent to that of the set of supporting polyhedra $\Pi(\Delta) = \{\Delta_1, \ldots, \Delta_r\}$, and therefore this data is often also called a nef partition.

It can be shown that given a nef partition $\Pi(\Delta)$ the polytopes[7]

$$\nabla_l = \left\langle \{0\} \cup E_l \right\rangle \ \subset N_{\mathbb{R}} \tag{3.15}$$

define again a nef partition $\Pi^*(\nabla) = \{\nabla_1, \ldots, \nabla_r\}$ such that the Minkowski sum $\nabla = \nabla_1 + \cdots + \nabla_r$ is a reflexive polytope. This is the combinatorial manifestation of mirror symmetry in terms of dual pairs of nef partitions of Δ^* and ∇^*, which we summarize in the diagram

$$\begin{array}{|c|ccc|c|}
\hline
\Delta = \Delta_1 + \ldots + \Delta_r & & & & \Delta^* = \left\langle \nabla_1, \ldots, \nabla_r \right\rangle \\
 & & & & \\
M_{\mathbb{R}} & & \begin{array}{c}\text{Mirror}\\ \text{Symmetry}\end{array} & & N_{\mathbb{R}} \\
 & & & & \\
\nabla^* = \left\langle \Delta_1, \ldots, \Delta_r \right\rangle & & (\Delta_l, \nabla_{l'}) \geq -\delta_{l\,l'} & & \nabla = \nabla_1 + \ldots + \nabla_r \\
\hline
\end{array} . \tag{3.16}$$

In the horizontal direction, we have the duality between the lattices M and N and mirror symmetry goes from the upper right to the lower left. The other diagonal has also a meaning in terms of mirror symmetry as we will explain below. The complete intersections $Y \subset \mathbb{P}_{\Delta^*}$ and $Y^* \subset \mathbb{P}_{\nabla^*}$ associated to the dual nef partitions are then mirror Calabi-Yau varieties.

Let us now apply the Batyrev-Borisov construction to the complete intersection eq. (3.1), hence $r = 2$. There exist several nef-partitions of Δ^*. The one which has the correct degrees $(3,1,0)$ and $(0,1,3)$ is, up to exchange of t_0 and t_1, $E_1 = \{\rho_i \,|i = 1, \ldots, 4\}$ and $E_2 = \{\rho_i \,|i = 5, \ldots, 8\}$. Adding the origin and taking the convex hull yields the polytopes

$$\nabla_1 = \left\langle \rho_1, \ldots, \rho_4, 0 \right\rangle, \qquad\qquad \nabla_2 = \left\langle \rho_5, \ldots, \rho_8, 0 \right\rangle, \tag{3.17}$$

[7] The brackets $\langle \cdots \rangle$ denote the convex hull.

where the ρ_i are defined in eq. (3.6). The two divisors cutting out the Calabi-Yau threefold are, according to eq. (3.11),

$$D_{0,1} = \sum_{i=1}^{4} D_i, \quad D_{0,2} = \sum_{i=5}^{8} D_i \qquad \Rightarrow \quad \widetilde{X} = D_{0,1} \cap D_{0,2} \ \subset \mathbb{P}_{\Delta^*} \tag{3.18}$$

Note that, while Δ^* has no further lattice points, its dual Δ has 18 vertices and 300 lattice points. Using the computer package PALP [35], we determine the associated polytopes Δ_1 and Δ_2 of the global sections of $\mathcal{O}(D_{0,1})$ and $\mathcal{O}(D_{0,2})$, respectively. In an appropriate lattice basis there is, up to symmetry, a unique nef partition consisting of

$$\Delta_1 = \big\langle \nu_1, \ldots, \nu_6, 0 \big\rangle, \qquad\qquad \Delta_2 = \big\langle \nu_7, \ldots, \nu_{12}, 0 \big\rangle, \tag{3.19}$$

where

$$\begin{aligned}
\nu_1 &= 2e_1 - e_2, & \nu_2 &= -e_1 + 2e_2, & \nu_3 &= -e_1 - e_2, \\
\nu_4 &= 2e_1 - e_2 - e_3, & \nu_5 &= -e_1 + 2e_2 - e_3, & \nu_6 &= -e_1 - e_2 - e_3, \\
\nu_7 &= 2e_4 - e_5, & \nu_8 &= -e_4 + 2e_5, & \nu_9 &= -e_4 - e_5, \\
\nu_{10} &= e_3 + 2e_4 - e_5, & \nu_{11} &= e_3 - e_4 + 2e_5, & \nu_{12} &= e_3 - e_4 - e_5.
\end{aligned} \tag{3.20}$$

Among these 12 vectors there are the 7 independent linear relations

$$\begin{aligned}
3\nu_3 + \nu_4 + \nu_5 - 2\nu_6 &= 0, & 3\nu_9 + \nu_{10} + \nu_{11} - 2\nu_{12} &= 0, \\
\nu_1 - \nu_3 - \nu_4 + \nu_6 &= 0, & -\nu_1 + \nu_2 + \nu_4 - \nu_5 &= 0, \\
\nu_7 - \nu_9 - \nu_{10} + \nu_{12} &= 0, & -\nu_7 + \nu_8 + \nu_{10} - \nu_{11} &= 0, \\
-\nu_2 + \nu_5 - \nu_8 + \nu_{11} &= 0.
\end{aligned} \tag{3.21}$$

The convex hull $\nabla^* = \langle \Delta_1, \Delta_2 \rangle$ yields the fan $\Sigma(\nabla^*)$ and, consequently, the toric variety $\mathbb{P}_{\nabla^*}$. Let $D_i^*, i = 1, \ldots, 12$ be the divisors associated to the vertices ν_i. Then, by eq. (3.11), the nef partition eq. (3.19) defines the divisors

$$D_{0,1}^* = \sum_{i=1}^{6} D_i^*, \quad D_{0,2}^* = \sum_{i=7}^{12} D_i^*, \qquad \Rightarrow \quad \widetilde{X}^* = D_{0,1}^* \cap D_{0,2}^* \ \subset \mathbb{P}_{\nabla^*} \tag{3.22}$$

cutting out the mirror complete intersection $\widetilde{X}^*$. In contrast to Δ^*, the polytope ∇^* contains extra integral points. We find that it contains, in addition to the origin and the vertices in eq. (3.20), the 26 points

$$\begin{aligned}
\nu_{13} &= \frac{1}{3}(\nu_4 + \nu_5 + \nu_6) = -e_3, & \nu_{12+6k+i+j} &= \frac{1}{3}(\nu_{3k+i} + 2\nu_{3k+j}), \\
\nu_{14} &= \frac{1}{3}(\nu_{10} + \nu_{11} + \nu_{12}) = e_3, & \nu_{15+6k+i+j} &= \frac{1}{3}(\nu_{3k+j} + 2\nu_{3k+i}) \\
&\forall\ k \in \{0, \ldots, 3\},\ (i,j) \in \big\{(1,2),(1,3),(2,3)\big\}.
\end{aligned} \tag{3.23}$$

For completeness, note that the dual polytope ∇ has 15 vertices and 24 lattice points. Running PALP to compute the Hodge numbers using the formula of [36], we obtain

$$h^{1,1}\big(\widetilde{X}\big) = h^{1,2}\big(\widetilde{X}\big) = h^{1,1}\big(\widetilde{X}^*\big) = h^{1,2}\big(\widetilde{X}^*\big) = 19, \tag{3.24}$$

in agreement with Part A [1], eq. (2.5).

So far, we have mainly focused on the information contained in the reflexive polytopes Δ^* and ∇^* and ignored their duals. We have already mentioned that in the reflexive case a generic section of $K^{-1}_{\mathbb{P}_{\Delta^*}}$ defines a Calabi-Yau manifold, and that such sections are provided by the lattice points of Δ. In other words, Δ and ∇ are the Newton polytopes of Y and Y^*, respectively. That is, the complete intersection Y (Y^*) is defined by r polynomial equations, and the exponents of the monomials in each equation are the lattice points in Δ (∇). More precisely, the Minkowski sum for, say, $\Delta = \Delta_1 + \cdots + \Delta_r$ defines r homogeneous polynomials

$$F_l(z) = \sum_{\substack{m\in \\ \Delta_l \cap M}} a_{l,m} \prod_{l'=1}^{r} \prod_{\substack{\rho_i \in \\ \nabla_{l'} \cap N}} z_i^{\langle m,\rho_i\rangle + \delta_{l\,l'}}, \qquad l = 1, \ldots, r \tag{3.25}$$

with coefficients $a_{l,m} \in \mathbb{C}$. The simultaneous vanishing of $F_1, \ldots, F_r$ then defines the complete intersection Calabi-Yau manifold $Y \subset \mathbb{P}_{\Delta^*}$. Exchanging Δ_l and $\nabla_{l'}$ in eq. (3.25) yields the equations F_l^* defining the mirror manifold Y^*. It is in this sense that the map from the upper left to the lower right in eq. (3.16) is also a manifestation of mirror symmetry. Since we will not need the actual polynomials for $\widetilde{X}$ and $\widetilde{X}^*$, we refrain from writing them explicitly. Instead, we refer the reader to section 4, where we determine the equations in a simpler situation.

3.3 Toric intersection ring

Up to now we have only considered one of the ingredients in the fan Σ, namely, the generators $\rho \in \Sigma^{(1)}$ which defined the $\mathbb{C}^\times$ action in eq. (3.3). The second ingredient is the exceptional set $Z(\Sigma)$. It corresponds to fixed loci of a continuous subgroup of $(\mathbb{C}^\times)^h$ for which the quotient eq. (3.3) is not well defined. Therefore, these loci have to be removed. In terms of the homogeneous coordinates z_i, this happens precisely when a subset $\{z_i \,|\, i \in I\}$, $I \subseteq \{1, \ldots, n\}$, of the coordinates vanishes simultaneously such that there is no cone $\sigma \in \Sigma$ containing all of the $\rho_i \subseteq \sigma$, $i \in I$. Hence, the set $Z(\Sigma)$ is the union of the sets $Z_I = \{[z_1 : \cdots : z_n] \,|\, z_i = 0 \,\forall i \in I\}$. Minimal index sets I with this property are called primitive collections [37]. In order to determine the index sets I we need a coherent[8] triangulation $T = T(\Delta^*)$ of the polytope Δ^* for which all simplices contain the origin. Different triangulations will yield different exceptional sets and, hence, different toric varieties. However, for simplicity, we will mostly suppress the choice of a triangulation in the notation. In the case of complete intersections, only those triangulations of Δ^* are compatible with a given nef partition that can be lifted to a triangulation of the corresponding Gorenstein cone, see [38].

The polytope defining projective space $\mathbb{P}^d$ admits a unique triangulation with the required properties, and this triangulation consists of $n = d + 1$ simplices. The only primitive collection is $I = \{1, \ldots, n\}$. This is well-known from the definition of projective space, where we have to remove the origin $z_1 = \cdots = z_{d+1} = 0$ from $\mathbb{C}^{d+1}$. Similarly, the

[8] Coherent triangulations, sometimes also called regular triangulations, satisfy some technical property that is equivalent to the associated toric quotient being Kähler.

JHEP10(2007)023

polyhedron Δ^* for the ambient space $\mathbb{P}_{\Delta^*}$ of $\widetilde{X}$ admits a unique triangulation, and the primitive collections are those of its factors, that is,

$$I_1 = \{1,2,3\}, \qquad I_2 = \{4,5\}, \qquad I_3 = \{6,7,8\}. \tag{3.26}$$

The mirror polyhedron ∇^*, on the other hand, admits a huge number of triangulations. We will discuss particularly interesting triangulations of the mirror polyhedron at the end of appendix A.

The primitive collections determine the cohomology ring of toric varieties and, together with the nef partition, complete intersections. Recall that if the collection $\rho_{i_1}, \dots, \rho_{i_k}$ of rays is not contained in at least one cone, then the corresponding homogeneous coordinates z_{i_l} are not allowed to vanish simultaneously. Therefore, the corresponding divisors D_{i_l} have no common intersection. Hence, we obtain non-linear relations $R_I = D_{i_1} \cdot \ldots \cdot D_{i_k} = 0$ in the intersection ring. It can be shown that all such relations are generated by the primitive collections $I = \{i_1, \dots, i_k\}$ defined above. The ideal generated by these R_I is called Stanley-Reisner ideal

$$I_{\mathrm{SR}} = \big\langle R_I,\, I \text{ primitive collection}\big\rangle \quad \subset \mathbb{Z}[D_1, \dots, D_n], \tag{3.27}$$

and $\mathbb{Z}[D_1, \dots, D_n]/I_{\mathrm{SR}}$ is the Stanley-Reisner ring. The intersection ring of a non-singular compact toric variety $\mathbb{P}_\Sigma$ is [39]

$$H^*\big(\mathbb{P}_\Sigma, \mathbb{Z}\big) = \mathbb{Z}\left[D_1, \dots, D_n\right] \Big/ \big\langle I_{\mathrm{SR}},\, \sum_i (m, \rho_i) D_i \big\rangle. \tag{3.28}$$

In other words, the intersection ring can be obtained from the Stanley-Reisner ring by adding the linear relations $\sum_i (m, \rho_i) D_i = 0$, where it is sufficient to take a set of basis vectors for $m \in M$. In particular, the intersection number of the divisors spanning a maximal-dimensional simplicial cone $\sigma = \mathrm{span}_{\mathbb{R}_\geq}\{\rho_{i_1}, \dots, \rho_{i_d}\}$ is

$$D_{i_1} \cdot \ldots \cdot D_{i_d} = \frac{1}{\mathrm{Vol}(\sigma)}, \tag{3.29}$$

where $\mathrm{Vol}(\sigma)$ is the lattice-volume, that is, the geometric volume divided by the volume $\frac{1}{d!}$ of a basic simplex. For practical purposes it is sufficient to compute one of these volumes, the remaining intersections can be obtained using the linear and non-linear relations.

Having found the intersection ring of the ambient toric variety, we now turn to the complete intersection $Y \subset \mathbb{P}_{\Delta^*}$. The toric part of its even-degree intersection ring is [40]

$$H^{\mathrm{ev}}_{\mathrm{toric}}\big(Y, \mathbb{Q}\big) = \mathbb{Q}\left[D_1, \dots, D_n\right] / I_Y, \tag{3.30}$$

where I_Y is the ideal quotient

$$I_Y = \Big\langle I_{\mathrm{SR}}, \sum_i (m, \rho_i) D_i \Big\rangle \, : \, \prod_{l=1}^{r} D_{0,l}. \tag{3.31}$$

Note that it can happen that some of the D_i appear as generators of I_Y. This means that they can be set to zero in the intersection ring. Geometrically, this means that these divisors do not intersect a generic complete intersection Y. While the intersection ring depends on the triangulation $T(\Delta^*)$ through the primitive collections defining the Stanley-Reisner ideal, we conjecture that the divisors D_i not intersecting Y are independent of the choice of triangulation. This conjecture is proven for $r = 1$ and supported by a large amount of empirical evidence for $r > 1$. We conclude that the dimension $\dim H^2_{\text{toric}}(Y)$ is in general smaller than $h^{1,1}(Y)$ for the following two reasons: Only $h = n - d = \dim H^2(\mathbb{P}_{\Delta^*}, \mathbb{Z})$ divisors are realized in the ambient toric variety $\mathbb{P}_{\Delta^*}$, and some of them may not descend to the complete intersection Y. Using the adjunction formula we can compute the the Chern classes of Y by expanding

$$\mathrm{c}(Y) = \frac{\prod_{i=1}^{n}(1 + D_i)}{\prod_{l=1}^{r}(1 + D_{0,l})}. \tag{3.32}$$

The intersection ring together with the second Chern class determine the diffeomorphism type of a simply-connected Calabi-Yau manifold [41]. If we consider the cohomology with integral coefficients there can be torsion and, in fact, this is what this paper is all about. Unfortunately, a combinatorial formula in terms of the fan $\Sigma(\Delta)$ for the torsion in the integral cohomology of a toric Calabi-Yau manifold is only known in the hypersurface case [6].

We now illustrate these concepts in the example of the complete intersection $\widetilde{X} \subset \mathbb{P}_{\Delta^*} = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$ and its mirror manifold $\widetilde{X}^*$. In eq. (3.26) we already determined the primitive collections, hence the corresponding Stanley-Reisner ideal is

$$I_{\text{SR}} = \big\langle D_1 D_2 D_3, D_4 D_5, D_6 D_7 D_8 \big\rangle. \tag{3.33}$$

The linear equivalences are $D_1 = D_2$, $D_1 = D_3$, $D_4 = D_5$, $D_6 = D_7$, $D_6 = D_8$ and, hence, we can choose $K_1 = D_4$, $K_2 = D_1$, $K_3 = D_6$ as a basis for $H^2(\mathbb{P}_{\Delta^*})$. In terms of this basis, we obtain $D_{0,1} = K_1 + 3K_2$ and $D_{0,2} = K_1 + 3K_3$, see eq. (3.11). Therefore, the ideal $I_{\widetilde{X}}$ in eq. (3.18) is

$$I_{\widetilde{X}} = \big\langle {K_3}^2 K_2 - {K_2}^2 K_3,\ K_1 K_2 - 3\,{K_2}^2,\ K_1 K_3 - 3\,{K_3}^2,\ {K_1}^2,\ {K_2}^3,\ {K_3}^3 \big\rangle\,. \tag{3.34}$$

Next, we define the restriction of the K_i to $\widetilde{X}$ to be the divisors

$$\tilde{J}_i = K_i \cdot \widetilde{X} = K_i (K_1 + 3K_2)(K_1 + 3K_3). \tag{3.35}$$

We need to compute one of the intersection numbers directly from the volume of a cone, say, $\tilde{J}_1 \tilde{J}_2 \tilde{J}_3 = K_1 K_2 K_3 (K_1 + 3K_2)(K_1 + 3K_3) = 9 K_1 K_2^2 K_3^2$, where we made use of the relations in $I_{\widetilde{X}}$. Using eq. (3.29), this intersection can be evaluated to be

$$\begin{aligned} 9 K_1 K_2^2 K_3^2 &= 9 D_1 D_2 D_4 D_6 D_7 = 9/\operatorname{Vol}\big(\langle \rho_1, \rho_2, \rho_4, \rho_6, \rho_7 \rangle\big) \\ &= 9/\operatorname{Vol}\big(\langle e_1, e_2, e_3, e_4, e_5 \rangle\big) = 9. \end{aligned} \tag{3.36}$$

Then, again using eq. (3.34), we see that the only non-vanishing intersection numbers and the second Chern class are

$$\begin{gathered} \tilde{J}_2^2 \tilde{J}_3 = 3, \quad \tilde{J}_1 \tilde{J}_2 \tilde{J}_3 = 9, \quad \tilde{J}_2 \tilde{J}_3^2 = 3, \\ \mathrm{c}_2\big(\widetilde{X}\big) \cdot \tilde{J}_1 = 0, \quad \mathrm{c}_2\big(\widetilde{X}\big) \cdot \tilde{J}_2 = 36, \quad \mathrm{c}_2\big(\widetilde{X}\big) \cdot \tilde{J}_3 = 36. \end{gathered} \tag{3.37}$$

Note that only $h^{1,1}_{\text{toric}}(\widetilde{X}) = 3$ of the $h^{1,1}(\widetilde{X}) = 19$ parameters are realized torically. Comparing the triple intersection numbers with eq. (2.8), it is clear that these 3 toric divisors are precisely the G-invariant divisors on $\widetilde{X}$.

A similar, though much more complicated, calculation can be done for $\widetilde{X}^* \subset \mathbb{P}_{\nabla^*}$. Using the results of appendix A one can show that, among the points in eq. (3.23), the 14 divisors $D^*_{13}, D^*_{14}, D^*_{12+6k+i+j}, D^*_{15+6k+i+j}$, $k = 0, 2$ appear as generators of eq. (3.31) and, therefore, do not intersect $\widetilde{X}^*$. Subtracting from the remaining 24 divisors in eqs. (3.20) and (3.23) the remaining 5 linear relations in eq. (3.21), we find that all $h^{1,1}_{\text{toric}}(\widetilde{X}^*) = h^{1,1}(\widetilde{X}^*) = 19$ moduli are realized torically.

3.4 Mori cone

As we have just seen, the cohomology classes D_i span $H^2(\mathbb{P}_\Sigma, \mathbb{R}) = H^{1,1}(\mathbb{P}_\Sigma)$. The Kähler classes of a smooth projective toric variety $\mathbb{P}_\Sigma$ form an open cone in $H^{1,1}(\mathbb{P}_\Sigma)$ called the Kähler cone $\mathcal{K}(\mathbb{P}_\Sigma)$. This cone has a combinatorial description in terms of the fan Σ, which we now review.

First, define a support function to be a continuous function $\psi : N_\mathbb{R} \to \mathbb{R}$ given on each cone $\sigma \in \Sigma$ by an $m_\sigma \in M_\mathbb{R}$ via

$$\psi(\rho) = (m_\sigma, \rho) \quad \forall \rho \in \sigma \subset N_\mathbb{R}. \tag{3.38}$$

A support function determines a divisor $D = \sum_i \psi(\rho_i) D_i$. We say that D is convex if ψ is a convex function on $N_\mathbb{R}$. The convex classes form a non-empty strongly convex polyhedral cone in $H^{1,1}(\mathbb{P}_\Sigma)$ whose interior is the Kähler cone $\mathcal{K}(\mathbb{P}_\Sigma)$. Such a support function is strictly convex if and only if

$$\psi(\rho_{i_1} + \cdots + \rho_{i_k}) > \psi(\rho_{i_1}) + \cdots + \psi(\rho_{i_k}) \tag{3.39}$$

for every primitive collection $I = \{i_1, \ldots, i_k\}$ [40]. The dual of the Kähler cone $\mathcal{K}(\mathbb{P}_\Sigma)$ is called the Mori cone or the cone of numerically effective curves $\mathrm{NE}(\mathbb{P}_\Sigma)$. Its generators can be described by vectors $l^{(a)}$ of the corresponding linear relations $\sum_i l^{(a)}_i \rho_i = 0$. Each face of the Kähler cone $\mathcal{K}(\mathbb{P}_\Sigma)$ is dual to an edge of $\mathrm{NE}(\mathbb{P}_\Sigma)$. These edges are generated by curves $c^{(a)}$, and the entries of the vector $l^{(a)}$ are

$$\left(l^{(a)}\right)_i = c^{(a)} \cdot D_i. \tag{3.40}$$

A practical algorithm to find the generators for $l^{(a)}$ in terms of the triangulation $T(\Delta^*)$ is described in [42]. Of course, we are not interested in the ambient space but in a complete intersection $Y \subset \mathbb{P}_{\Delta^*}$. The restriction of a Kähler class on the ambient space yields a Kähler class on Y, but not every Kähler class on Y arises that way. We define the toric part of the Kähler cone on Y as the restriction [43]

$$\mathcal{K}(Y)_{\text{toric}} = \mathcal{K}(\mathbb{P}_\Sigma)\big|_Y \subset \mathcal{K}(Y). \tag{3.41}$$

In the simplicial case, we can always take the basis J_i of $H^2_{\text{toric}}(Y, \mathbb{Q})$ to be edges of the Kähler cone. The dual of the toric Kähler cone of Y is the (toric) Mori cone $\mathrm{NE}(Y)_{\text{toric}}$.

This is sufficient for mirror symmetry purposes, however, it can be larger than the actual cone of effective curves. Once the generators $l^{(a)}$ of $\mathrm{NE}(\mathbb{P}_{\Delta^*})$ are determined, we need to add the information about the nef partition. For this purpose, we define

$$l_{0,m}^{(a)} \stackrel{\text{def}}{=} -D_{0,m} \cdot c^{(a)} \quad m = 1, \dots, r. \tag{3.42}$$

Finally, it is customary to write the generators of the Mori cone $\mathrm{NE}(Y)_{\text{toric}}$ as

$$l^{(a)} = \left(l_{0,1}^{(a)}, \dots, l_{0,r}^{(a)}; l_1^{(a)}, \dots, l_n^{(a)}\right), \tag{3.43}$$

which are, by abuse of notation, again denoted by $l^{(a)}$. The knowledge of the (toric) Mori cone is important for several reasons. It defines the local coordinates on the complex structure moduli space of the mirror Y^* near the point of maximal unipotent monodromy. Moreover, the generators enter the coefficients of the fundamental period which is a solution of the Picard-Fuchs equations as we will review in subsection 3.5.

For example, using the unique primitive collections in eq. (3.26), the Mori cone for $\mathbb{P}_{\Delta^*}$ is generated[9] by

$$\begin{aligned} l^{(1)} =& (0,\ 0,\ 0,\ 1,\ 1,\ 0,\ 0,\ 0) \\ l^{(2)} =& (1,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0) \\ l^{(3)} =& (0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 1). \end{aligned} \tag{3.44}$$

Recalling the nef partition $D_{0,1} = D_1 + \cdots + D_4$, $D_{0,2} = D_5 + \cdots + D_8$, we prepend $(-D_{0,1} \cdot c^{(a)}, -D_{0,2} \cdot c^{(a)}) = (-3, 0)$, $(-1, -1)$, $(0, -3)$, $a = 1, 2, 3$, to obtain the generators

$$\begin{aligned} l^{(1)} =& (-1, -1;\ 0,\ 0,\ 0,\ 1,\ 1,\ 0,\ 0,\ 0) \\ l^{(2)} =& (-3,\ \ 0;\ 1,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0) \\ l^{(3)} =& (\ \ 0, -3;\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 1) \end{aligned} \tag{3.45}$$

of the Mori cone $\mathrm{NE}(\widetilde{X})_{\text{toric}}$. Due to the large number of toric moduli, the calculation for the Mori cone $\mathrm{NE}(\mathbb{P}_{\nabla^*})$ of the ambient toric variety of the mirror $\widetilde{X}^*$ is much more complex.

3.5 B-model prepotential

Mirror symmetry identifies the quantum corrected Kähler moduli space of Y with the classical complex structure moduli space of Y^*, see the excellent treatise in [43] for details. The deformations of the complex structure of Y^* are encoded in the periods $\varpi = \int_\gamma \Omega$ and the latter can be computed from the equations F_l^* that cut out $Y^* \subset \mathbb{P}_{\nabla^*}$. Given the Mori cone eq. (3.43) and the classical intersections numbers $\kappa_{abc} = J_a \cdot J_b \cdot J_c$ we follow [43–45, 38] to write down a local expansion of the periods, convergent near the large complex structure point, which is characterized by its maximal unipotent monodromy. In the following, we will review just the bare essentials.

[9] We sort the Mori cone generators such that the first one corresponds to the $\mathbb{P}^1$ of the ambient space, and the second and third generator are the hyperplane sections of the two $\mathbb{P}^2$. In other words, we have $\tilde{J}_a \cdot c^{(b)} = \delta_a^b$. This is the basis of curves that we used for the A-model computation.

The coefficients a_i in the polynomial constraints F_l^* of the complete intersection Y^*, see eq. (3.25), define the complex structure of Y^*. A particular set of local coordinates u_a on the complex structure moduli space on Y^* is defined by

$$u_b = \prod_{m=1}^{r} a_{m,0}^{l_{0,m}^{(b)}} \prod_{i=1}^{n} a_i^{l_i^{(b)}} \qquad b = 1, \ldots, h \tag{3.46}$$

where $h \stackrel{\text{def}}{=} h^{1,1}_{\text{toric}}(Y)$ and $a_{m,0}$ is the coefficient in (3.25) corresponding to the origin in ∇_l. In these coordinates, the point of maximal unipotent monodromy is at $u_b = 0$. We define the cohomology-valued period

$$\varpi(u, J) = \sum_{\{n_a \geq = 0\}} \frac{\prod_{m=1}^{r} \left(1 - \sum_{a=1}^{h} l_{0,m}^{(a)} J_a\right)_{-\sum_{a=1}^{h} l_{0,m}^{(a)} n_a}}{\prod_{i=1}^{n} \left(1 + \sum_{a=1}^{h} l_i^{(a)} J_a\right)_{\sum_{a=1}^{h} l_i^{(a)} n_a}} \prod_{a=1}^{h} u_a^{n_a + J_a}. \tag{3.47}$$

where $(x)_n = \Gamma(x+n)/\Gamma(x)$ is the Pochhammer symbol. Note that the choice of triangulation is implicit in the generators $l^{(a)}$ of the Mori cone. Expanding $\varpi(u, J)$ by cohomology degree yields

$$\varpi(u, J) = \varpi^{(0)}(u) + \sum_{a=1}^{h} \varpi_a^{(1)}(u) J_a + \sum_{a=1}^{h} \varpi_a^{(2)}(u) \kappa_{abc} J_b J_c - \varpi^{(3)}(u)\, \mathrm{dVol}, \tag{3.48}$$

where dVol is the volume form. The coefficients in eq. (3.48) are the fundamental period $\varpi^{(0)}(u)$, that is, the unique solution to the Picard-Fuchs equations holomorphic at $u_a = 0$, and

$$\varpi_a^{(1)}(u) = \partial_{J_a} \varpi(u, J)|_{J=0}, \qquad \varpi_a^{(2)}(u) = \frac{1}{2} \kappa_{abc} \partial_{J_b} \partial_{J_c} \varpi(u, J)|_{J=0},$$
$$\varpi^{(3)}(u) = -\frac{1}{6} \kappa_{abc} \partial_{J_a} \partial_{J_b} \partial_{J_c} \varpi(u, J)|_{J=0}. \tag{3.49}$$

These coefficients coincide with the basis of solutions of the Picard- Fuchs equations obtained from the Frobenius method in [46, 31]. The B-model prepotential $\mathcal{F}_{Y^*,0}^B$ is

$$\mathcal{F}_{Y^*,0}^B(u) = \frac{1}{2\varpi^{(0)}(u)^2} \left(\varpi^{(0)}(u) \varpi^{(3)}(u) + \sum_{a=1}^{h} \varpi_a^{(1)}(u) \varpi_a^{(2)}(u) \right). \tag{3.50}$$

At the large complex structure point the mirror map defines natural flat coordinates on the Kähler moduli space of the original manifold Y, which are

$$t_i = \frac{\varpi_i^{(1)}(u)}{\varpi_0(u)}, \qquad i = 1, \ldots, h. \tag{3.51}$$

We also define $q_j = e^{2\pi i t_j} = u_j + O(u^2)$. One way to obtain the prepotential is to compute its third derivatives

$$C_{abc}^* = D_a D_b D_c \mathcal{F}_{Y^*,0}^B = \int_{Y^*} \Omega \wedge \partial_a \partial_b \partial_c \Omega, \tag{3.52}$$

and apply the Picard-Fuchs operators. This leads to linear differential equations, which determine C^*_{abc} up to a common constant, see again [46, 43] for details. The quantum corrected three point function $C_{ijk}(q)$ on Y follows from $C^*_{abc}(u)$ using the inverse mirror map eq. (3.51) $u = u(t)$, and one obtains

$$C_{ijk}(q) = \frac{1}{\varpi^{(0)}(u(q))^2} \frac{\partial u_a}{\partial t_i} \frac{\partial u_b}{\partial t_j} \frac{\partial u_c}{\partial t_k} C^*_{abc}(u(q)). \tag{3.53}$$

In practice, we use the formula

$$C_{ijk}(q) = \partial_{t_i} \partial_{t_j} \frac{\varpi^{(2)}_k(u(q))}{\varpi^{(0)}(u(q))}. \tag{3.54}$$

Integrating three times with respect to t_i yields the prepotential $\mathcal{F}^B_{Y^*,0}(t)$ up to a polynomial of degree three in t_i which can be determined partially by the topological data of Y.

Mirror symmetry then ensures that the B-model prepotential, eq. (3.50), is equal to the A-model prepotential. That is,

$$\mathcal{F}_{Y,0}(q) = \mathcal{F}^B_{Y^*,0}(u(q)). \tag{3.55}$$

This allows us to compute the instanton numbers n_d. For the case of interest,

$$\widetilde{X} \in \mathbb{P}_{\Delta^*} = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2, \tag{3.56}$$

we refer to [28] where this program been carried out in detail. The same calculation can in principle be done on the mirror X^*, but the large number of toric moduli again makes it highly extensive. Instead, we refer to the next section where a suitable quotient of $\widetilde{X}^*$ will be treated in detail for which the computations are reasonably simple.

4. Quotienting the B-Model

In this section we consider the quotient $X = \widetilde{X}/G$ in terms of toric geometry and study the mirror of X in this context. In order to achieve this, we first analyze the partial quotient $\overline{X} = \widetilde{X}/G_1$. Using the techniques introduced in section 3, we construct the mirror $\overline{X}^*$. Using their toric realization, we perform the B-model computation for the non-perturbative prepotentials $\mathcal{F}^{\mathrm{np}}_{\overline{X},0}$ and $\mathcal{F}^{\mathrm{np}}_{\overline{X}^*,0}$, respectively. Finally, we explain how one can implement the quotient by G_2 on both sides in order to obtain X and X^*.

4.1 The quotient by $\mathbf{G_1}$

We start with a review of the general discussion of free quotients of complete intersections in toric geometry in [31]. Consider a fan $\Sigma \subset N_{\mathbb{R}}$ and pick a lattice refinement $\bar{N}$ such that $\Gamma = \bar{N}/N$ is a finite abelian group. Such a lattice refinement consists of a finite sequence of lattice refinements of the form $N \to N + w_p\mathbb{Z}$ which are described by a vector $w_p = \frac{1}{k_p}\sum \alpha_{pi}\rho_i$ with $\alpha_{pi} \in \mathbb{Z}$. The group Γ is then isomorphic to $\prod_p \mathbb{Z}_{k_p}$. Let $\bar{\Sigma}$ be the fan obtained from Σ by relating everything to the lattice $\bar{N}$. In this context, we make some additional identifications in the toric quotient eq. (3.3) [47]. One finds that $V_{\bar{\Sigma}} = V_\Sigma/\Gamma$ is the quotient of V_Σ by the finite abelian group Γ. Its action on the homogeneous coordinates is by multiplication by phases

$$\big[z_1 : \cdots : z_n\big] \mapsto \big[\xi^{\alpha_1} z_1 : \cdots : \xi^{\alpha_n} z_n\big], \qquad \xi = e^{\frac{2\pi \mathrm{i}}{k}}, \tag{4.1}$$

for every cyclic subgroup of order k. We will denote such group actions by $\mathbb{Z}_k : (\alpha_1, \ldots, \alpha_n)$. If V_Σ is a compact toric variety, then the quotient $V_{\bar\Sigma}$ is never free [39]. However, a hypersurface or complete intersection in V_Σ need not intersect the set of fixed points, and in that case we get a smooth quotient manifold with nontrivial fundamental group.

We now apply this to $\mathbb{P}_{\Delta^*} = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$ defined in eq. (3.6). The first step in performing the quotient of $\mathbb{P}_{\Delta^*}$ by G_1 thus amounts to a refinement $\bar N = w\,\mathbb{Z} + N$ of the lattice N with index $|G_1| = 3$. From the definition eq. (2.3a) of the action of G_1 on $\mathbb{P}_{\Delta^*}$ and eq. (3.8) we read off that the refinement is by a vector

$$w \in \frac{1}{3}\big(\rho_2 + 2\rho_3 + \rho_7 + 2\rho_8\big) + \mathbb{Z}^5. \tag{4.2}$$

The resulting polytope $\bar\Delta^*$ admits the same nef partition as Δ^* in eq. (3.17),

$$\bar\nabla_1 = \langle \bar\rho_1, \ldots, \bar\rho_4, 0\rangle, \qquad\qquad \bar\nabla_2 = \langle \bar\rho_5, \ldots, \bar\rho_8, 0\rangle. \tag{4.3}$$

where we express the generators $\bar\rho$ in terms of ρ as

$$\begin{aligned} &\bar\rho_i = \rho_i, && i = 1, \ldots, 6\,, \\ &\bar\rho_7 = \rho_7 + e_1 + 2e_2 + e_4 + 2e_5, && \bar\rho_8 = \rho_8 - e_1 - 2e_2 - e_4 - 2e_5. \end{aligned} \tag{4.4}$$

It is easy to check that the $\bar\rho_i$ satisfy the same linear relations eq. (3.7) as the ρ_i, and that $w = \frac{1}{3}(\bar\rho_1 - \bar\rho_2 + \bar\rho_6 - \bar\rho_7) = -e_2 - e_5$. The $\bar\rho_i$ together with w therefore indeed generate the lattice $\bar N$. Note that, while all 8 non-zero lattice points of $\bar\Delta^*$ are vertices, the dual polytope $\bar\Delta$ has 18 vertices and 102 points. Using PALP [35] again, we compute the lattice points of the polytope $\bar\nabla^* = \langle \bar\Delta_1, \bar\Delta_2\rangle \subset M_{\mathbb{R}}$, which will describe the ambient space of the mirror $\overline{X}^*$ of $\overline{X}$. We find

$$\bar\Delta_1 = \langle \bar\nu_1, \ldots, \bar\nu_6, 0\rangle, \quad \bar\Delta_2 = \langle \bar\nu_7, \ldots, \bar\nu_{12}, 0\rangle, \tag{4.5}$$

where we express the vertices $\bar\nu_i$ in terms of the vertices ν_i of ∇^* as

$$\bar\nu_{3k+1} = \nu_{3k+1}, \quad \bar\nu_{3k+2} = \nu_{3k+2} - e_5, \quad \bar\nu_{3k+3} = \nu_{3k+3} + e_5, \quad k = 0, \ldots, 3. \tag{4.6}$$

Again, it is easy to check that the $\bar\nu_i$ satisfy the same linear relations eq. (3.21) as the ν_i. It turns out that the lattice points of $\bar\nabla^*$ generate a sublattice $\bar M$ of index 3 in M, and the lattice refinement is generated by

$$w^* = \frac{1}{3}\big(\bar\nu_1 + 2\bar\nu_2 + 2\bar\nu_7 + \bar\nu_8\big) = e_2 + e_4 - e_5. \tag{4.7}$$

Among the points of ∇^* listed in eq. (3.23) only ν_{13} and ν_{14} are also lattice points of the sublattice $\bar{M}$. In fact, we have $\bar{\nu}_{13} = \nu_{13}$ and $\bar{\nu}_{14} = \nu_{14}$. Hence, $\bar{\nabla}^*$ has 12 vertices and 15 lattice points; its dual $\bar{\nabla} = \bar{\nabla}_1 + \bar{\nabla}_2$ has 42 lattice points among which 15 are vertices.[10]

Once we have the polytopes $\bar{\Delta}^*$ and $\bar{\nabla}^*$, we can construct $\overline{X}$ and $\overline{X}^*$ as complete intersections entirely analogous to $\widetilde{X}$ and $\widetilde{X}^*$, see section 3. That is, using eq. (3.11), we define

$$\overline{X} = \bar{D}_{0,1} \cap \bar{D}_{0,2}, \quad \overline{X}^* = \bar{D}^*_{0,1} \cap \bar{D}^*_{0,2} \tag{4.9}$$

in terms of the nef partitions eq. (4.3) and (4.5), respectively. Here, $\bar{D}_i$ and $\bar{D}^*_i$ denote the divisors associated to the generators $\bar{\rho}_i$ and $\bar{\nu}_i$, respectively. The absence of fixed points of the G_1 action on the complete intersection $\widetilde{X}$ is guaranteed by the fact that the resulting polytope $\bar{\Delta}^* \subset \bar{N}_{\mathbb{R}}$ has no additional lattice points [31]. Hence, $\overline{X} = \widetilde{X}/G_1$ has a non-trivial fundamental group $\pi_1(\overline{X}) = \mathbb{Z}_3$. Surprisingly, it turns out that the mirror $\overline{X}^*$ is a free quotient as well. To see this recall that, as noticed above, the lattice points of $\bar{\nabla}^*$ generate a sublattice $\bar{M}$ of index 3 in M. Furthermore, $\bar{\nabla}^*$ also has no additional lattice points with respect to ∇^*. Therefore, there is a group $G_1^* \simeq \mathbb{Z}_3$ acting torically on $\mathbb{P}_{\nabla^*}$. On the homogeneous coordinates this action is

$$g_1^* : \big[z_1 : \cdots : z_{12}\big] \mapsto \big[\zeta z_1 : \zeta^2 z_2 : z_3 : \cdots : z_6 : \zeta^2 z_7 : \zeta z_8 : z_9 : \cdots : z_{12}\big]. \tag{4.10}$$

Hence, $\overline{X}^* = \widetilde{X}^*/G_1^*$ also has a non-trivial fundamental group $\pi_1(\overline{X}^*) = \mathbb{Z}_3$. Note that this never happens for hypersurfaces in toric varieties [6]. Having the toric representation of $\overline{X}$ and $\overline{X}^*$, we can now compute their Hodge numbers. It turns out that

$$h^{1,1}\big(\overline{X}\big) = h^{1,2}\big(\overline{X}\big) = h^{1,1}\big(\overline{X}^*\big) = h^{1,2}\big(\overline{X}^*\big) = 7, \tag{4.11}$$

in agreement with Part A [1], eq. (5.16).

4.2 The quotient by $\mathbf{G_2}$

We now turn to the G_2 action, which does not act torically. Hence, we cannot, in principle, find a toric variety containing $X = \overline{X}/G_2$ as we did for the G_1 quotient above. However, at least we have to ensure that $\overline{X}$ and $\overline{X}^*$ are G_2-symmetric. This can be achieved via suitable symmetries in the toric data.

The easy part of the toric data for $\overline{X}$ is the polytope $\bar{\Delta}^*$. The G_2 action on the ambient space permutes the homogeneous coordinates, see eq. (2.3b). In terms of toric geometry,

[10] Note that all of our polytopes differ from the non-free $\mathbb{Z}_3 \times \mathbb{Z}_3$ quotient of Δ^* defined in [28], Proposition 7.1. In the notation of [31] their quotient is

$$\nabla^* \neq \mathbb{P}\begin{pmatrix} 1\,1\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,1\,1\,1\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1\,1 \end{pmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 1 & 1 \end{bmatrix} \Bigg/ \begin{matrix} \mathbb{Z}_3 : \; 0\,1\,2\,0\,0\,0\,0\,0 \\ \mathbb{Z}_3 : \; 0\,0\,0\,0\,1\,2\,0\,0 \end{matrix} \tag{4.8}$$

and has 21 points and 8 vertices in the lattice N.

JHEP10(2007)023

this means that it permutes the corresponding points of the polytope. That is,[11]

$$
\begin{aligned}
&g_2 : \bar\rho_i \mapsto \bar\rho_{1+(i \bmod 3)} \quad \forall i \in \{1,2,3\},\\
&g_2 : \bar\rho_4 \mapsto \bar\rho_4, \quad \bar\rho_5 \mapsto \bar\rho_5,\\
&g_2 : \bar\rho_{5+i} \mapsto \bar\rho_{6+(i \bmod 3)} \quad \forall i \in \{1,2,3\}.
\end{aligned} \tag{4.12}
$$

It induces a mirror group action G_2^* on $\overline{X}^*$ which is geometrical, rather than a quantum symmetry as discussed in [48]. The action of G_2^* is obviously the dual group action on the dual lattice M, which again must be a symmetry of the relevant polytope $\bar\nabla^*$. We find that

$$
g_2^* : \bar\nu_{3k+i} \mapsto \bar\nu_{3k+1+(i \bmod 3)} \quad \forall k = 0, \ldots, 3,\ i \in \{1,2,3\}. \tag{4.13}
$$

As a check on the mirror group action, note that the matrix of scalar products, see eq. (4.15) below, is invariant. That is,

$$
\big\langle g_2(\bar\rho_l),\ g_2^*(\bar\nu_{l'})\big\rangle = \big\langle \bar\rho_l,\ \bar\nu_{l'}\big\rangle \quad \forall\, l,\, l'. \tag{4.14}
$$

By abuse of notation, we denote the corresponding cyclic permutation of homogeneous coordinates by g_2^* as well. Using this action, we define the mirror of X to be $X^* = \overline{X}^*/G_2^*$. This idea has already been used for the construction of mirrors of orbifolds of the quintic [49] soon after the discovery of the first mirror construction by Greene and Plesser.

Following eq. (3.25), the equations for the Calabi-Yau complete intersections $\overline{X}$ and $\overline{X}^*$ are defined by evaluating the matrix of scalar products $\langle \bar\rho_i, \bar\nu_j \rangle + \delta_{l\,l'}$, which are

$$
\begin{array}{c|ccccccc|ccccccc}
\langle\,,\rangle + \delta_{l\,l'} & \bar\nu_1 & \bar\nu_2 & \bar\nu_3 & \bar\nu_4 & \bar\nu_5 & \bar\nu_6 & \bar\nu_{13} & \bar\nu_7 & \bar\nu_8 & \bar\nu_9 & \bar\nu_{10} & \bar\nu_{11} & \bar\nu_{12} & \bar\nu_{14} \\
\hline
\bar\rho_1 & 3 & 0 & 0 & 3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar\rho_2 & 0 & 3 & 0 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar\rho_3 & 0 & 0 & 3 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar\rho_4 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hline
\bar\rho_5 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\bar\rho_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 3 & 0 & 0 & 1 \\
\bar\rho_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 3 & 0 & 1 \\
\bar\rho_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 3 & 1
\end{array} \tag{4.15}
$$

The equations of X can now be read off from the columns of eq. (4.15), and one finds

$$
F_1 = (\lambda_5 t_0 + \lambda_6 t_1)(x_0^3 + x_1^3 + x_2^3) + (\lambda_7 t_0 + \lambda_8 t_1) x_0 x_1 x_2, \tag{4.16a}
$$

$$
F_2 = (\lambda_1 t_0 + \lambda_4 t_1)(y_0^3 + y_1^3 + y_2^3) + (\lambda_2 t_0 + \lambda_3 t_1) y_0 y_1 y_2, \tag{4.16b}
$$

where the G_2-symmetry has been imposed. Note that the last monomial in each equation corresponds to the vector $0 \in \bar\Delta_l$, $l = 1, 2$. Two of the eight coefficients λ_m can be fixed by normalizing the equations, say $\lambda_4 = \lambda_5 = 1$, and three correspond to the symmetries of $\mathbb{P}^1$, that is, SL(2) transformations of $[t_0 : t_1]$. Hence, we can, for example, set $\lambda_6 = \lambda_7 = \lambda_8 =$

[11] We define the modulus operation such that $(i \bmod 3) \in \{0, 1, 2\}$.

0. This leaves us with 3 complex structure deformations λ_1, λ_2, and λ_3, see eqs. (2.2a) and (2.2b).

The equations defining X^* correspond to the rows of eq. (4.15), that is,

$$F_1^* = a_1(z_1^3 z_4^3 + z_2^3 z_5^3 + z_3^3 z_6^3)z_{13} + (a_2 z_{10} z_{11} z_{12} z_{14} + a_3 z_4 z_5 z_6 z_{13}) z_1 z_2 z_3, \tag{4.17a}$$

$$F_2^* = a_4(z_7^3 z_{10}^3 + z_8^3 z_{11}^3 + z_9^3 z_{12}^3)z_{14} + (a_5 z_4 z_5 z_6 z_{13} + a_6 z_{10} z_{11} z_{12} z_{14}) z_7 z_8 z_9, \tag{4.17b}$$

where, again, invariance under G_2^* has been imposed and the last monomial of each equation comes from the lattice point $0 \in \bar{\nabla}_l$, $, l = 1, 2$. Both equations are homogeneous with respect to all seven scaling degrees that follow from the linear relations eq. (3.21). Among the twelve scalings of the coordinates z_i, six are compatible with the cyclic permutations g_2^*, see eq. (4.13). Subtracting the three G_2 symmetric independent scalings among the relations eq. (3.21), there remains one torus action that acts effectively on the parameters plus two normalizations of the equations. As expected, the six parameters a_m of the equations of X^* thus become the 3 complex structure moduli.

So far, we only considered the polytopes $\bar{\Delta}^*$ and $\bar{\nabla}^*$. However, this is only part of the toric data defining the manifolds $\overline{X}$ and $\overline{X}^*$, respectively. In addition, we need the triangulations and the corresponding exceptional sets. A change in the triangulation corresponds to a flop of the toric variety. The very real danger is that not all, and perhaps none, of the flopped Calabi-Yau manifolds are G_2-symmetric. For $\overline{X} \subset \mathbb{P}_{\bar{\Delta}^*}$ this turns out to be unproblematic, but for $\overline{X}^* \subset \mathbb{P}_{\bar{\nabla}^*}$ we will find a condition for the choice of a triangulation.

4.3 B-model on $\overline{\mathbf{X}}$

We now return to the discussion of the triangulations and the intersection ring of $\overline{X}$. The analogous, but technically much more involved discussion of $\overline{X}^*$ will be presented in subsection 4.5.

For $\overline{X}$ everything is straightforward since the G_1-quotient did not introduce additional lattice points in the associated polytope $\bar{\Delta}^*$. Therefore, just like for the polytope Δ^* of the covering space $\widetilde{X}$, there exists a unique triangulation. In particular the primitive collections, the Stanley-Reisner ideal, and the ideal $I_{\overline{X}}$ are identical to the ones in eqs. (3.26), (3.33), and (3.34) since they are derived from the same triangulation. Moreover, one can easily see that this triangulation is G_2-invariant and, hence, $\overline{X}$ is G_2 symmetric.

The only change is in the normalization of the intersection ring in eq. (3.36), since the total volume has to be divided by $3 = |G_1|$. This can also be seen in eq. (4.4), where the volume of the cone is now 3 instead of 1. Hence, on $\overline{X}$ the intersection ring and the second Chern class are

$$\begin{aligned} &\bar{J}_2^2 \bar{J}_3 = 1, && \bar{J}_1 \bar{J}_2 \bar{J}_3 = 3, && \bar{J}_2 \bar{J}_3^2 = 1, \\ &c_2\left(\overline{X}\right) \cdot \bar{J}_1 = 0, && c_2\left(\overline{X}\right) \cdot \bar{J}_2 = 12, && c_2\left(\overline{X}\right) \cdot \bar{J}_3 = 12. \end{aligned} \tag{4.18}$$

Comparing these intersection numbers with eq. (2.8), it is clear that the toric divisors should be identified with the G_1-invariant divisors on $\overline{X}$ as

$$\bar{J}_1 = \phi, \quad \bar{J}_2 = \tau_1, \quad \bar{J}_3 = \tau_2. \tag{4.19}$$

The curves spanning the Mori cone on the cover turn out to be G_1-invariant as well. Therefore, the Mori cones $\mathrm{NE}(\mathbb{P}_{\bar{\Delta}^*})$ and $\mathrm{NE}(\overline{X})_{\text{toric}}$ are identical to those in eqs. (3.44) and (3.45), respectively.

Following the steps given in section 3 we now want to compute the B-model prepotential $\mathcal{F}^B_{\overline{X}^*,0}$, plug in the mirror map, and obtain the prepotential on $\overline{X}$

$$\mathcal{F}^{\text{np}}_{\overline{X},0}(P,Q_1,Q_2,Q_3,R_1,R_2,R_3,b_1). \tag{4.20}$$

We immediately realize the following two caveats:

- We do not know how to incorporate the torsion curves $H_2(\overline{X},\mathbb{Z})_{\text{tors}} = \mathbb{Z}_3$ into the toric mirror symmetry calculation.
- Of the 7 Kähler classes on $\overline{X}$, only 3 are toric.

This means that only 3 out of the $7+1$ variables in the prepotential are accessible, and the remaining ones are set to one. Looking at the intersection numbers eq. (4.18), it is clear that the 3 divisors are precisely the G_2-invariant divisors on $\overline{X}$, see eq. (2.8). Therefore, these 3 variables must be those that map to the variables p, q, and r on X. By comparing with eq. (2.11), we see that the corresponding variables on $\overline{X}$ are P, Q_1, and R_1. Hence, we actually only compute

$$\mathcal{F}^{\text{np}}_{\overline{X},0}(P,Q_1,1,1,R_1,1,1,1) = \sum_{n_1,n_2,n_3} n^{\overline{X}}_{(n_1,n_2,n_3)} \operatorname{Li}_3\left(P^{n_1}Q_1^{n_2}R_1^{n_3}\right). \tag{4.21}$$

In effect, this means that the resulting instanton numbers are not just the instantons in a single integral homology class, but the instanton numbers in a whole set of integral homology classes. The instanton numbers sum over all curve classes that cannot be distinguished by $P,Q_1,R_1 \in \operatorname{Hom}\left(H_2(\overline{X},\mathbb{Z}),\mathbb{C}^\times\right)$. Up to total degree 4 and the symmetry

$$n^{\overline{X}}_{(n_1,n_2,n_3)} = n^{\overline{X}}_{(n_1,n_3,n_2)}, \tag{4.22}$$

the resulting instanton numbers are

$$\begin{array}{llll} n^{\overline{X}}_{(1,0,0)} = 27 & n^{\overline{X}}_{(1,0,1)} = 108 & n^{\overline{X}}_{(1,0,2)} = 378 & n^{\overline{X}}_{(1,0,3)} = 1080 \\ n^{\overline{X}}_{(1,1,1)} = 432 & n^{\overline{X}}_{(1,1,2)} = 1512 & n^{\overline{X}}_{(2,0,1)} = -54 & n^{\overline{X}}_{(2,0,2)} = -756 \\ n^{\overline{X}}_{(2,1,1)} = 864 & n^{\overline{X}}_{(3,0,1)} = 9. & & \end{array} \tag{4.23}$$

4.4 Instanton numbers of X

Knowing the prepotential on $\overline{X}$, we now want to divide out the free G_2 action and arrive at the prepotential on X. Since we do not know the complete expansion but only eq. (4.21), we have to set $b_1 = b_2 = 1$ in the descent equation (2.11). This yields

$$\begin{aligned} \mathcal{F}^{\text{np}}_{X,0}(p,q,r,1,1) &= \frac{1}{3}\mathcal{F}^{\text{np}}_{\overline{X},0}\big(p,\, q,\, 1,\, 1,\, r,\, 1,\, 1,\, 1\big) \\ &= \sum_{n_1,n_2,n_3} n^{X}_{(n_1,n_2,n_3)} \operatorname{Li}_3\left(p^{n_1}q^{n_2}r^{n_3}\right). \end{aligned} \tag{4.24}$$

Up to the symmetry $n^X_{(n_1,n_2,n_3)} = n^X_{(n_1,n_3,n_2)}$, the non-vanishing instanton numbers for X up to total degree 5 are

$$\begin{array}{llll} n^X_{(1,0,0)} = 9 & n^X_{(1,0,1)} = 36 & n^X_{(1,0,2)} = 126 & n^X_{(1,0,3)} = 360 \\ n^X_{(1,0,4)} = 945 & n^X_{(1,1,1)} = 144 & n^X_{(1,1,2)} = 504 & n^X_{(1,1,3)} = 1440 \\ n^X_{(1,2,2)} = 1764 & n^X_{(2,0,1)} = -18 & n^X_{(2,0,2)} = -252 & n^X_{(2,0,3)} = -1728 \\ n^X_{(2,1,1)} = 288 & n^X_{(2,1,2)} = 3960 & n^X_{(3,0,1)} = 3 & n^X_{(3,0,2)} = 252 \\ n^X_{(3,1,1)} = 756, & & & \end{array} \tag{4.25}$$

Unfortunately, this direct calculation misses the torsion information and only yields the expansion $\mathcal{F}^{\mathrm{np}}_{X,0}(p,q,r,1,1)$. The b_1 dependence was lost because the toric methods do not yield this part, and the b_2 dependence was lost because the relevant divisor on $\overline{X}$ was not toric. Comparing with the full expansion of the prepotential

$$\mathcal{F}^{\mathrm{np}}_{X,0}(p,q,r,b_1,b_2) = \sum_{\substack{n_1,n_2,n_3 \\ m_1,m_2}} n^X_{(n_1,n_2,n_3,m_1,m_2)} \operatorname{Li}_3\left(p^{n_1} q^{n_2} r^{n_3} b_1^{m_1} b_2^{m_2}\right), \tag{4.26}$$

see Part A eq. (8.39), this means we only obtain the sum of the instanton numbers over all torsion classes

$$n^X_{(n_1,n_2,n_3)} = \sum_{m_1,m_2=0}^{2} n^X_{(n_1,n_2,n_3,m_1,m_2)}. \tag{4.27}$$

Clearly, this destroys the torsion information, that is, the instanton numbers $n^X_{(n_1,n_2,n_3)}$ do not depend on the torsion part of the integral homology. For comparison purposes, we list the instanton numbers $n^X_{(n_1,n_2,n_3)}$ for $0 \leq n_1, n_2, n_3 \leq 5$ in table 2.

4.5 B-model on $\overline{\mathbf{X}}^*$

We now study the mirror $\overline{X}^*$, which sits in a more complicated ambient toric variety. Consequently, the analysis is more involved. The big advantage, however, will turn out to be that all $h^{11}(\overline{X}^*) = 7$ Kähler moduli are toric, which will enable us to obtain the full instanton expansion.

Since the polytope $\bar{\nabla}^*$ in eq. (4.6) is not simplicial, we have to specify a resolution of the singularities, that is, a triangulation $T(\bar{\nabla}^*)$. Moreover, not any triangulation will do, but we have to make sure that it is compatible with the action of the permutation group G_2^*. While a tedious technicality, the existence of such a resolution has to be shown in order to establish the existence of a geometrical mirror family of X. In particular, we show in appendix A that there is no projective resolution of the ambient space among the 720 coherent star triangulations of $\bar{\nabla}^*$ that respects the permutation symmetry eq. (4.13). In other words, if one demands G_2^* symmetry then the ambient toric variety cannot be chosen to be Kähler, but only a complex manifold. Clearly, in that case there is no Kähler cone and the usual toric mirror symmetry algorithm does not work. What comes to the rescue is that there are two classes of non-symmetric projective resolutions for which the symmetry-violating exceptional sets do not intersect $\overline{X}^*$. Hence the complete intersection is G_2-symmetric, even though the ambient space is not.

JHEP10(2007)023

$n^X_{(0,n_2,n_3)}$

$n_2 \backslash n_3$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0

$n^X_{(1,n_2,n_3)}$

$n_2 \backslash n_3$	0	1	2	3	4	5
0	9	36	126	360	945	2268
1	36	144	504	1440	3780	9072
2	126	504	1764	5040	13230	31752
3	360	1440	5040	14400	37800	90720
4	945	3780	13230	37800	99225	238140
5	2268	9072	31752	90720	238140	571536

$n^X_{(2,n_2,n_3)}$

$n_2 \backslash n_3$	0	1	2	3	4	5
0	0	−18	−252	−1728	−9000	−38808
1	−18	288	3960	27648	143748	620928
2	−252	3960	54432	380160	1976472	8537760
3	−1728	27648	380160	2654208	13799808	59609088
4	−9000	143748	1976472	13799808	71748000	309920688
5	−38808	620928	8537760	59609088	309920688	1338720768

$n^X_{(3,n_2,n_3)}$

$n_2 \backslash n_3$	0	1	2	3	4	5
0	0	3	252	4158	40173	287415
1	3	756	15390	164280	1259685	7763364
2	252	15390	426708	5427684	46537092	310465062
3	4158	164280	5427684	73971360	657552966	4487097816
4	40173	1259685	46537092	657552966	5948103483	41016575313
5	287415	7763364	310465062	4487097816	41016575313	284581389204

$n^X_{(4,n_2,n_3)}$

$n_2 \backslash n_3$	0	1	2	3	4	5
0	0	0	−144	−6048	−107280	−1235520
1	0	−306	−12348	−207000	−2273400	−19066500
2	−144	−12348	348480	14609520	235219680	2505155400
3	−6048	−207000	14609520	520226784	8245864800	87989812560
4	−107280	−2273400	235219680	8245864800	131759049600	1417949658000
5	−1235520	−19066500	2505155400	87989812560	1417949658000	15365394415800

$n^X_{(5,n_2,n_3)}$

$n_2 \backslash n_3$	0	1	2	3	4	5
0	0	0	45	5670	189990	3508920
1	0	36	13140	474840	8793648	111499020
2	45	13140	1112886	38961252	777759975	10723515300
3	5670	474840	38961252	1952428464	47357606430	732897531720
4	189990	8793648	777759975	47357606430	1237373786439	19911043749420
5	3508920	111499020	10723515300	732897531720	19911043749420	327006066948660

Table 2: Summed instanton numbers $n^X_{(n_1,n_2,n_3)} = \sum_{m_1,m_2} n^X_{(n_1,n_2,n_3,m_1,m_2)}$ (hence not distinguishing torsion) computed by mirror symmetry. The table contains all non-vanishing instanton numbers for $0 \leq n_1, n_2, n_3 \leq 6$.

We conclude that the extended Kähler moduli space of $\overline{X}^*$ contains two symmetric phases. We will denote these two classes of triangulations by $T_\pm = T_\pm(\bar\nabla^*)$, see appendix A. In fact, the two phases are topologically distinct, and only the triangulation T_+ describes the threefold $\overline{X}^*$ that we are interested in. In appendix B, we will investigate the other triangulation T_- which describes a flop of $\overline{X}^*$.

Following subsection 3.3, given the triangulation T_+, we can determine the primitive collections. This immediately yields the Stanley-Reisner ideal

$$\begin{aligned} I_{\mathrm{SR}} = \Big\langle & \bar D_1\bar D_{13}, \bar D_2\bar D_4, \bar D_2\bar D_{13}, \bar D_3\bar D_4, \bar D_3\bar D_5, \bar D_3\bar D_{13}, \bar D_4\bar D_{10}, \bar D_4\bar D_{11}, \bar D_4\bar D_{12}, \\ & D_4\bar D_{14}, \bar D_5\bar D_{10}, \bar D_5\bar D_{11}, \bar D_5\bar D_{12}, \bar D_5\bar D_{14}, \bar D_6\bar D_{10}, \bar D_6\bar D_{11}, \bar D_6\bar D_{12}, \\ & \bar D_6\bar D_{14}, \bar D_{13}\bar D_{10}, \bar D_{13}\bar D_{11}, \bar D_{13}\bar D_{12}, \bar D_{13}\bar D_{14}, \bar D_7\bar D_{14}, \bar D_8\bar D_{10}, \bar D_8\bar D_{12}, \\ & \bar D_8\bar D_{14}, \bar D_9 D_{10}, \bar D_9\bar D_{14}, \bar D_1\bar D_2\bar D_3, \bar D_1\bar D_2\bar D_6, D_1\bar D_5\bar D_6, \bar D_4\bar D_5\bar D_6, \\ & \bar D_7\bar D_8\bar D_9, \bar D_7\bar D_9\bar D_{11}, \bar D_7\bar D_{11}\bar D_{12}, \bar D_{10}\bar D_{11}\bar D_{12}\Big\rangle \end{aligned} \tag{4.28}$$

where we dropped the superscript $*$ on $\bar D$ for ease of notation. From this, in turn, we obtain the generators $\bar l_+^{(a)}$ of the Mori cone $\mathrm{NE}(\mathbb{P}_{\bar\nabla^*})$:

$$\begin{array}{l} \bar l_+^{(1)} = (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 1,\ \ 0,\ \ 0,-1,\ \ 0,\ \ 0,\ \ 0,\ \ 1) \\ \bar l_+^{(2)} = (\ \ 1,\ \ 0,\ \ 0,-1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 1,\ \ 0) \\ \bar l_+^{(3)} = (-1,\ \ 1,\ \ 0,\ \ 1,-1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0) \\ \bar l_+^{(4)} = (\ \ 0,-1,\ \ 1,\ \ 0,\ \ 1,-1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0) \\ \bar l_+^{(5)} = (\ \ 0,\ \ 0,-1,\ \ 0,\ \ 0,\ \ 1,\ \ 0,-1,\ \ 0,\ \ 0,\ \ 1,\ \ 0,\ \ 0,\ \ 0) \\ \bar l_+^{(6)} = (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,-1,\ \ 0,\ \ 1,\ \ 1,\ \ 0,-1,\ \ 0,\ \ 0) \\ \bar l_+^{(7)} = (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 1,-1,\ \ 0,-1,\ \ 1,\ \ 0,\ \ 0) \\ \bar l_+^{(8)} = (\ \ 0,\ \ 0,\ \ 0,\ \ 1,\ \ 1,\ \ 1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,-3,\ \ 0) \\ \bar l_+^{(9)} = (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 1,\ \ 1,\ \ 1,\ \ 0,-3). \end{array} \tag{4.29}$$

A dual basis for the generators of the Kähler cone $\mathcal{K}(\mathbb{P}_{\bar\nabla^*})$ is

$$\begin{aligned} \bar K_1 &= \bar D_{13} + 2\,\bar D_1 - \bar D_2 - \bar D_3 + \bar D_9 + \bar D_7 + \bar D_8 + 3\,\bar D_4, \\ \bar K_2 &= 3\,\bar D_1 + \bar D_{13} + 3\,\bar D_4, \\ \bar K_3 &= \bar D_{13} + 2\,\bar D_1 + 3\,\bar D_4, \\ \bar K_4 &= \bar D_{13} + 2\,\bar D_1 - \bar D_2 + 3\,\bar D_4, \\ \bar K_5 &= \bar D_{13} + 2\,\bar D_1 - \bar D_2 - \bar D_3 + 3\,\bar D_4, \\ \bar K_6 &= \bar D_{13} + 2\,\bar D_1 - \bar D_2 - \bar D_3 + \bar D_9 + \bar D_8 + 3\,\bar D_4, \\ \bar K_7 &= \bar D_8 + \bar D_{13} + 2\,\bar D_1 - \bar D_2 - \bar D_3 + 3\,\bar D_4, \\ \bar K_8 &= \bar D_4 + \bar D_1, \\ \bar K_9 &= \bar D_{10} + \bar D_7. \end{aligned} \tag{4.30}$$

The Calabi-Yau complete intersection $\overline{X}^*$ is then defined by $\overline{X}^* = \bar{K}_1\bar{K}_2$. It turns out that the divisors $\bar{D}_{13}$, $\bar{D}_{14}$ do not intersect $\overline{X}^*$. Therefore, all

$$h^{1,1}_{\text{toric}}\big(\overline{X}^*\big) = h^{1,1}\big(\overline{X}^*\big) = 7 \tag{4.31}$$

Kähler moduli are realized torically. Since there are two divisors that do not intersect, finding the Mori cone is somewhat subtle. First, we have to restrict the lattice of linear relations to the sublattice orthogonal to these two directions. For the generators of the toric Mori cone $\mathrm{NE}(\overline{X}^*)_{\text{toric}}$, this means that $\bar{l}^{(1)}_+ \to 3\bar{l}^{(1)}_+ + \bar{l}^{(9)}_+$, $\bar{l}^{(2)}_+ \to 3\bar{l}^{(2)}_+ + \bar{l}^{(8)}_+$ and that we drop $\bar{l}^{(8)}_+$, $\bar{l}^{(9)}_+$ as well as the entries corresponding to intersections with $\bar{D}_{13}$, $\bar{D}_{14}$. In addition, we prepend the intersection numbers with $\bar{D}_{0,1}$ and $\bar{D}_{0,2}$. This yields

$$\begin{aligned}
\bar{l}^{(1)}_+ &= (-3,\ 0;\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 3,\ 0,\ 0,-2,\ 1,\ 1)\\
\bar{l}^{(2)}_+ &= (\ 0,-3;\ 3,\ 0,\ 0,-2,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)\\
\bar{l}^{(3)}_+ &= (\ 0,\ 0;-1,\ 1,\ 0,\ 1,-1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)\\
\bar{l}^{(4)}_+ &= (\ 0,\ 0;\ 0,-1,\ 1,\ 0,\ 1,-1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)\\
\bar{l}^{(5)}_+ &= (\ 0,\ 0;\ 0,\ 0,-1,\ 0,\ 0,\ 1,\ 0,-1,\ 0,\ 0,\ 1,\ 0)\\
\bar{l}^{(6)}_+ &= (\ 0,\ 0;\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,-1,\ 0,\ 1,\ 1,\ 0,-1)\\
\bar{l}^{(7)}_+ &= (\ 0,\ 0;\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,-1,\ 0,-1,\ 1).
\end{aligned} \tag{4.32}$$

The dual basis of divisors is

$$\begin{aligned}
\bar{J}^*_1 &= \frac{1}{3}\bar{K}_1^2\bar{K}_2, & \bar{J}^*_2 &= \frac{1}{3}\bar{K}_1\bar{K}_2^2, & \bar{J}^*_5 &= \bar{K}_1\bar{K}_2\bar{K}_5,\\
\bar{J}^*_3 &= \bar{K}_1\bar{K}_2\bar{K}_3, & \bar{J}^*_4 &= \bar{K}_1\bar{K}_2\bar{K}_4, & &\\
\bar{J}^*_6 &= \bar{K}_1\bar{K}_2\bar{K}_6, & \bar{J}^*_7 &= \bar{K}_1\bar{K}_2\bar{K}_7. & &
\end{aligned} \tag{4.33}$$

We now try to identify this basis $\bar{J}^*_1, \dots, \bar{J}^*_7$ of divisors on $\overline{X}^*$ with the basis $\{\phi, \tau_1, \upsilon_1, \psi_1, \tau_2, \upsilon_2, \psi_2\}$ of divisors on $\overline{X}$ in eq. (2.5). It turns out that there is more than one way to identify the bases if one only wants to preserve the triple intersection numbers. To obtain a unique answer, we also need to identify the actions by G_2^* and G_2 as well. First, the G_2^* action on $H^2(\mathbb{P}_{\bar{\nabla}^*}, \mathbb{Z})$ is defined by eq. (4.13). Using the linear equivalence relations

$$\begin{aligned}
2\bar{D}_1 - \bar{D}_2 - \bar{D}_3 + 2\bar{D}_4 - \bar{D}_5 - \bar{D}_6 &= 0\\
-\bar{D}_1 + 2\bar{D}_2 - \bar{D}_3 - \bar{D}_4 + 2\bar{D}_5 - \bar{D}_6 &= 0\\
2\bar{D}_7 - \bar{D}_8 - \bar{D}_9 + 2\bar{D}_{10} - \bar{D}_{11} - \bar{D}_{12} &= 0\\
-\bar{D}_2 + \bar{D}_3 - \bar{D}_5 + \bar{D}_6 - \bar{D}_8 + \bar{D}_9 - \bar{D}_{11} + \bar{D}_{12} &= 0\\
-\bar{D}_4 - \bar{D}_5 - \bar{D}_6 + \bar{D}_{10} + \bar{D}_{11} + \bar{D}_{12} - \bar{D}_{13} + \bar{D}_{14} &= 0
\end{aligned} \tag{4.34}$$

and the definition eq. (4.33), one can compute the induced group action on $H^2(\overline{X}^*, \mathbb{Z})$. We

JHEP10(2007)023

find

$$g_2^* \begin{pmatrix} \bar{J}_1^* \\ \bar{J}_2^* \\ \bar{J}_3^* \\ \bar{J}_4^* \\ \bar{J}_5^* \\ \bar{J}_6^* \\ \bar{J}_7^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \bar{J}_1^* \\ \bar{J}_2^* \\ \bar{J}_3^* \\ \bar{J}_4^* \\ \bar{J}_5^* \\ \bar{J}_6^* \\ \bar{J}_7^* \end{pmatrix}. \tag{4.35}$$

Second, recall that the G_2 action on the divisors of $\overline{X}^*$ is

$$g_2 \begin{pmatrix} \phi \\ \tau_1 \\ \upsilon_1 \\ \psi_1 \\ \tau_2 \\ \upsilon_2 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & -1 & 0 & 0 & 0 \\ 0 & 3 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} \phi \\ \tau_1 \\ \upsilon_1 \\ \psi_1 \\ \tau_2 \\ \upsilon_2 \\ \psi_2 \end{pmatrix}, \tag{4.36}$$

see Part A eq. (8.54).

The essentially unique[12] identification of divisors on $\overline{X}^*$ and $\overline{X}$ then turns out to be

$$\begin{aligned} &\bar{J}_1^* = \tau_1, && \bar{J}_2^* = \tau_2, && \bar{J}_3^* = \psi_2, && \bar{J}_4^* = \upsilon_2, \\ &\bar{J}_5^* = \phi, && \bar{J}_6^* = 3\tau_1 + \upsilon_1 - \psi_1 = g_2(\psi_1), && \bar{J}_7^* = \upsilon_1. \end{aligned} \tag{4.37}$$

Note that we are identifying divisors on $\overline{X}^*$ with divisors on $\overline{X}$ in eq.(4.37), something that one would usually not do. However, in view of the anticipated self-mirror property, $\overline{X}^* \cong \overline{X}$, this is a sensible thing to try to attempt. And, indeed, the identification above is an isomorphism of the intersection rings.

Regardless of this identification, we now continue to apply mirror symmetry. First, the second Chern class is

$$\begin{aligned} \mathrm{c}_2\left(\overline{X}^*\right) \cdot \bar{J}_1^* = 12, \qquad\qquad & \mathrm{c}_2\left(\overline{X}^*\right) \cdot \bar{J}_5^* = 0, \quad \mathrm{c}_2\left(\overline{X}^*\right) \cdot \bar{J}_2^* = 12, \\ \mathrm{c}_2\left(\overline{X}^*\right) \cdot \bar{J}_3^* = \mathrm{c}_2\left(\overline{X}^*\right) \cdot \bar{J}_6^* = 24, \quad & \mathrm{c}_2\left(\overline{X}^*\right) \cdot \bar{J}_4^* = \mathrm{c}_2\left(\overline{X}^*\right) \cdot \bar{J}_7^* = 12. \end{aligned} \tag{4.38}$$

Using this information, we now compute the B-model prepotential

$$\begin{aligned} \mathcal{F}^B_{\overline{X},0}(q_1, \ldots, q_7) = {} & 3q_5 + 3q_4q_5 + \frac{3}{8}q_5^2 + 3q_5q_7 + 3q_7q_5q_6 + 3q_4q_5q_7 + \frac{1}{9}q_5^3 \\ & + 3q_3q_4q_5 + \frac{3}{8}q_5^2q_7^2 + 3q_3q_4q_5q_7 + \frac{3}{8}q_4^2q_5^2 \\ & + 3q_4q_3q_2q_5 + 3q_7q_5q_4q_6 + 3q_1q_5q_6q_7 + \frac{3}{64}q_5^4 + O(q^5). \end{aligned} \tag{4.39}$$

Finally, we insert the mirror map and obtain the A-model prepotential on $\overline{X}^*$. Since we already identified the bases $\bar{J}_1^*, \ldots, \bar{J}_7^*$ with the divisors on $\overline{X}$, we will use the same names

[12] Up to the $\hat{g}_2$ and $g_2^{-1}\hat{g}_2$ symmetry.

(but with an added $*$ superscript) for the Fourier-transformed variables to expand the prepotential. With this notation, we obtain

$$\begin{aligned}\mathcal{F}^{\mathrm{np}}_{\overline{X}^*,0}(P^*,Q_1^*,Q_2^*,Q_3^*,R_1^*,R_2^*,R_3^*,1) &= 3P^* + \frac{3}{8}P^{*2} + \frac{1}{9}P^{*3} + \frac{3}{64}P^{*4} + \frac{3}{125}P^{*5} \\ &+ 3P^*Q_2^* + \frac{3}{8}P^{*2}Q_2^{*2} + 3P^*Q_2^*Q_3^* + 3P^*R_2^* + \frac{3}{8}P^{*2}R_2^{*2} + 3P^*R_2^*Q_2^* \\ &+ 3P^*R_2^*Q_2^*Q_3^* + 3P^*R_2^*R_3^* + 3P^*R_2^*R_3^*Q_2^* + 3P^*R_2^*R_3^*Q_2^*Q_3^* \\ &+ 3P^*Q_1^*R_2^*R_3^* + 3P^*Q_1^*R_2^*R_3^*Q_2^* + 9P^*Q_1^*R_2^*R_3^*R_3^* + 3P^*Q_2^*Q_3^*R_1^* \\ &+ 3P^*Q_2^*Q_3^*R_1^*R_2^* + 9P^*Q_2^*Q_3^*R_1^*Q_3^* + \big(\text{total degree} \geq 6\big),\end{aligned} \tag{4.40}$$

see also Part A eq. (8.64). The instanton numbers on $\overline{X}^*$ are the expansion coefficients

$$\begin{aligned}&\mathcal{F}^{\mathrm{np}}_{\overline{X}^*,0}(P^*,Q_1^*,Q_2^*,Q_3^*,R_1^*,R_2^*,R_3^*,1) \\ &\qquad = \sum_{n_1,\dots,n_7} n^{\overline{X}^*}_{(n_1,n_2,n_3,n_4,n_5,n_6,n_7)} \operatorname{Li}_3\left(P^{*n_1}Q_1^{*n_2}Q_2^{*n_3}Q_3^{*n_4}R_1^{*n_5}R_2^{*n_6}R_3^{*n_7}\right).\end{aligned} \tag{4.41}$$

We see that we almost get the complete instanton expansion eq. (2.10), we only miss the expansion in the b_1^* variable which is not computed by the toric mirror symmetry algorithm. Up to total degree 5, the instanton numbers are

$$\begin{array}{llll} n^{\overline{X}^*}_{(1,0,0,0,0,0,0)} = 3 & n^{\overline{X}^*}_{(1,0,0,0,0,1,0)} = 3 & n^{\overline{X}^*}_{(1,0,0,0,0,1,1)} = 3 & n^{\overline{X}^*}_{(1,0,1,0,0,0,0)} = 3 \\ n^{\overline{X}^*}_{(1,0,1,0,0,1,0)} = 3 & n^{\overline{X}^*}_{(1,0,1,0,0,1,1)} = 3 & n^{\overline{X}^*}_{(1,0,1,1,0,0,0)} = 3 & n^{\overline{X}^*}_{(1,0,1,1,0,1,0)} = 3 \\ n^{\overline{X}^*}_{(1,0,1,1,0,1,1)} = 3 & n^{\overline{X}^*}_{(1,1,0,0,0,1,2)} = 9 & n^{\overline{X}^*}_{(1,0,1,2,1,0,0)} = 9 & n^{\overline{X}^*}_{(1,0,1,1,1,1,0)} = 3 \\ n^{\overline{X}^*}_{(1,1,1,0,0,1,1)} = 3 & n^{\overline{X}^*}_{(1,1,0,0,0,1,1)} = 3 & n^{\overline{X}^*}_{(1,0,1,1,1,0,0)} = 3. & \end{array} \tag{4.42}$$

Finally, let us take a look at the G_2^* action, see eq. (4.13). Of the 7 generators of the toric Mori cone, eq. (4.32), only the 3 generators $\bar{l}_+^{(1)}$, $\bar{l}_+^{(2)}$ and $\bar{l}_+^{(5)}$ are invariant. Not surprisingly, the dual G_2^*-invariant divisors

$$\bar{J}_5^* = \phi, \quad \bar{J}_1^* = \tau_1, \quad \bar{J}_2^* = \tau_2 \tag{4.43}$$

were identified with the G_2-invariant divisors on $\overline{X}$ in eq. (4.37). Therefore, only 3 Kähler parameters survive to the quotient $X^* = \overline{X}^*/G_2^*$, and we have

$$h^{1,1}\big(X\big) = h^{1,2}\big(X\big) = h^{1,1}\big(X^*\big) = h^{1,2}\big(X^*\big) = 3. \tag{4.44}$$

4.6 Instanton numbers of $\mathbf{X}^*$

Now that we have the expression eq. (4.41) for the prepotential on $\overline{X}^*$, we can again apply a suitable variable substitution

$$\Big\{P^*,Q_1^*,Q_2^*,Q_3^*,R_1^*,R_2^*,R_3^*,b_1^*,b_2^*\Big\} \longrightarrow \Big\{p^*,q^*,r^*,b_1^*,b_2^*\Big\} \tag{4.45}$$

and obtain the prepotential on the quotient $X^* = \overline{X}^*/G_2^*$. The correct way to replace the variables is determined by the group action on the homology and cohomology as we

(n_1, n_2, n_3)	$n^{X^*}_{(n_1,n_2,n_3,0)}$	$n^{X^*}_{(n_1,n_2,n_3,1)}$	$n^{X^*}_{(n_1,n_2,n_3,2)}$	$n^{X}_{(n_1,n_2,n_3)}$
$(1, 0, 0)$	3	3	3	9
$(1, 0, 1)$	12	12	12	36
$(1, 0, 2)$	42	42	42	126
$(1, 0, 3)$	120	120	120	360
$(1, 1, 1)$	48	48	48	144
$(1, 1, 2)$	168	168	168	504
$(2, 0, 1)$	-6	-6	-6	-18
$(2, 0, 2)$	-84	-84	-84	-252
$(2, 1, 1)$	96	96	96	288
$(3, 0, 1)$	3	0	0	3

Table 3: Instanton numbers $n^{X^*}_{(n_1,n_2,n_3,m_2)}$ computed by toric mirror symmetry. They are invariant under the exchange $n_2 \leftrightarrow n_3$, so we only display them for $n_2 \leq n_3$.

explained in Part A. Having computed the G_2^*-action in eq. (4.35), we determine the descent equation for the prepotential to be[13]

$$\mathcal{F}^{\text{np}}_{X^*,0}\big(p^*, q^*, r^*, b_1^*, b_2^*\big) = \frac{1}{|G_2^*|}\mathcal{F}^{\text{np}}_{\overline{X}^*,0}\big(p^*, q^*, b_2^*, b_2^*, r^*, b_2^{*2}, b_2^{*2}, b_1^*\big). \tag{4.46}$$

Using the series expansion of the prepotential for $b_1^* = 1$ on $\overline{X}^*$ from subsection 4.5, we now find that

$$\begin{aligned}
\mathcal{F}^{\text{np}}_{X^*,0}(p^*, q^*, r^*, 1, b_2^*) = \sum_{j=0}^{2} 3 \times \Big(& \operatorname{Li}_3(p^* b_2^{*j}) + 4\operatorname{Li}_3(p^* q^* b_2^{*j}) + 4\operatorname{Li}_3(p^* r^* b_2^{*j}) \\
& + 14\operatorname{Li}_3(p^* q^{*2} b_2^{*j}) + 16\operatorname{Li}_3(p^* q^* r^* b_2^{*j}) + 14\operatorname{Li}_3(p^* r^{*2} b_2^{*j}) \\
& + 40\operatorname{Li}_3(p^* q^{*3} b_2^{*j}) + 56\operatorname{Li}_3(p^* q^{*2} r b_2^{*j}) + 56\operatorname{Li}_3(p^* q^* r^2 b_2^{*j}) \\
& + 40\operatorname{Li}_3(p^* r^{*3} b_2^{*j}) + 105\operatorname{Li}_3(p^* q^{*4} b_2^{*j}) + 160\operatorname{Li}_3(p^* q^{*3} r^* b_2^{*j}) \\
& - 2\operatorname{Li}_3(p^{*2} q^* b_2^{*j}) - 2\operatorname{Li}_3(p^{*2} r^* b_2^{*j}) \\
& - 28\operatorname{Li}_3(p^{*2} q^{*2} b_2^{*j}) + 32\operatorname{Li}_3(p^{*2} q^* r^* b_2^{*j}) - 28\operatorname{Li}_3(p^{*2} r^{*2} b_2^{*j})\Big) \\
& + 3\operatorname{Li}_3(p^{*3} q^*) + 3\operatorname{Li}_3(p^{*3} r^*) \\
& + \big(\text{total } p^*, q^*, r^*\text{-degree } \geq 5\big).
\end{aligned} \tag{4.47}$$

The corresponding instanton numbers

$$\mathcal{F}^{\text{np}}_{X^*,0}(p^*, q^*, r^*, 1, b_2^*) = \sum_{n_1,n_2,n_3,m_2} n^{X^*}_{(n_1,n_2,n_3,m_2)} \operatorname{Li}_3\big(p^{*n_1} q^{*n_2} r^{*n_3} b_2^{*m_2}\big) \tag{4.48}$$

are listed in table 3.

[13]Interestingly, eq. (4.46) turns out to be exactly analogous to eq. (2.11), even though the identification of divisors on $\overline{X}^*$ and $\overline{X}$ is not just a relabeling of divisors.

For comparison purposes, we list the summed instanton numbers on X as well, see eq. (4.27). One observes that the sum over the more refined instanton numbers on X^* equals the summed instanton number on X, another clue towards X being self-mirror.

4.7 Instanton numbers assuming the self-mirror property

So far, we have alluded to X being possibly self-mirror, but not actually made use of this property. Now we are going to assume the self-mirror property and, hence, obtain the prepotential on X as

$$\mathcal{F}^{\text{np}}_{X,0}(p,q,r,b_1,b_2) = \mathcal{F}^{\text{np}}_{X^*,0}(p,q,r,b_1,b_2). \tag{4.49}$$

Note that at linear and quadratic order in p we can actually recover the b_1, b_2 expansion from the summed instanton numbers in subsection 4.4 and the factorization which we will prove in section 6.

In contrast, for the prepotential terms at order p^3 we have to use the X^* prepotential to obtain the b_2 expansion from eq. (4.47). Since this is based on a toric computation on $\overline{X}^*$, we do not directly obtain the b_1 expansion. However, note that the fact that g_1 acted torically, eq. (2.3a), and g_2 non-torically, eq. (2.3b), is just a consequence of the choice of coordinate system on $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$. By a suitable coordinate choice, we could have made any one of the four $\mathbb{Z}_3$ subgroups of $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ act torically. Therefore, any combination of b_1, b_2 other than $1 = b_1^0 b_2^0$ has to occur in the same way in the complete series expansion of the prepotential. We conclude that the prepotential can only depend on b_1 and b_2 through the combinations

$$1, \qquad \sum_{i,j=0}^{2} b_1^i b_2^j. \tag{4.50}$$

This observation lets us recover the full b_1, b_2 expansion of the prepotential. To summarize, we obtain

$$\begin{aligned}
\mathcal{F}^{\text{np}}_{X*,0}(p,q,r,b_1,b_2) = \sum_{i,j=0}^{2} \Big(& \operatorname{Li}_3(pb_1^i b_2^j) + 4\operatorname{Li}_3(pqb_1^i b_2^j) + 4\operatorname{Li}_3(prb_1^i b_2^j) \\
& + 14\operatorname{Li}_3(pq^2b_1^i b_2^j) + 16\operatorname{Li}_3(pqrb_1^i b_2^j) + 14\operatorname{Li}_3(pr^2b_1^i b_2^j) \\
& + 40\operatorname{Li}_3(pq^3b_1^i b_2^j) + 56\operatorname{Li}_3(pq^2rb_1^i b_2^j) + 56\operatorname{Li}_3(pqr^2b_1^i b_2^j) \\
& + 40\operatorname{Li}_3(pr^3b_1^i b_2^j) + 105\operatorname{Li}_3(pq^4b_1^i b_2^j) + 160\operatorname{Li}_3(pq^3rb_1^i b_2^j) \\
& + 196\operatorname{Li}_3(pq^2r^2b_1^i b_2^j) + 160\operatorname{Li}_3(pqr^3b_1^i b_2^j) + 105\operatorname{Li}_3(pr^4b_1^i b_2^j) \\
& - 2\operatorname{Li}_3(p^2qb_1^i b_2^j) - 2\operatorname{Li}_3(p^2rb_1^i b_2^j) - 28\operatorname{Li}_3(p^2q^2b_1^i b_2^j) \\
& + 32\operatorname{Li}_3(p^2qrb_1^i b_2^j) - 28\operatorname{Li}_3(p^2r^2b_1^i b_2^j) - 192\operatorname{Li}_3(p^2q^3b_1^i b_2^j) \\
& + 440\operatorname{Li}_3(p^2q^2rb_1^i b_2^j) + 440\operatorname{Li}_3(p^2qr^2b_1^i b_2^j) - 192\operatorname{Li}_3(p^2r^3b_1^i b_2^j) \Big) \\
& + 3\operatorname{Li}_3(p^3q) + 3\operatorname{Li}_3(p^3r) + 9\operatorname{Li}_3(p^3q^2) + 27\sum_{i,j=0}^{2}\operatorname{Li}_3(p^3q^2b_1^i b_2^j) \\
& + 9\operatorname{Li}_3(p^3q^2) + 27\sum_{i,j=0}^{2}\operatorname{Li}_3(p^3q^2b_1^i b_2^j) \\
& + 27\operatorname{Li}_3(p^3qr) + 81\sum_{i,j=0}^{2}\operatorname{Li}_3(p^3qrb_1^i b_2^j) + \big(\text{total } p,q,r\text{-degree } \geq 6\big).
\end{aligned} \tag{4.51}$$

JHEP10(2007)023

$n^X_{(1,n_2,n_3,0,0)}$

$n_2 \backslash n_3$	0	1	2	3	4
0	1	4	14	40	105
1	4	16	56	160	
2	14	56	196		
3	40	160			
4	105				

$n^X_{(1,n_2,n_3,m_1,m_2)}$, $(m_1,m_2) \neq (0,0)$

$n_2 \backslash n_3$	0	1	2	3	4
0	1	4	14	40	105
1	4	16	56	160	
2	14	56	196		
3	40	160			
4	105				

$n^X_{(2,n_2,n_3,0,0)}$

$n_2 \backslash n_3$	0	1	2	3
0	0	-2	-28	-192
1	-2	32	440	
2	-28	440		
3	-192			

$n^X_{(2,n_2,n_3,m_1,m_2)}$, $(m_1,m_2) \neq (0,0)$

$n_2 \backslash n_3$	0	1	2	3
0	0	-2	-28	-192
1	-2	32	440	
2	-28	440		
3	-192			

$n^X_{(3,n_2,n_3,0,0)}$

$n_2 \backslash n_3$	0	1	2
0	0	**3**	**36**
1	**3**	**108**	
2	**36**		

$n^X_{(3,n_2,n_3,m_1,m_2)}$, $(m_1,m_2) \neq (0,0)$

$n_2 \backslash n_3$	0	1	2
0	0	**0**	**27**
1	**0**	**81**	
2	**27**		

Table 4: Instanton numbers $n^X_{(n_1,n_2,n_3,m_1,m_2)}$ computed by mirror symmetry. The table contains all non-vanishing instanton numbers for $n_1 + n_2 + n_3 \leq 5$. The entries marked in **bold** depend non-trivially on the torsion part of their respective homology class.

Obtaining all of these terms required a computation of $\mathcal{F}^B_{X,0}$ in eq. (4.39) up to total degree 23 in the 7 variables, which is close to the limit of what can be done with current desktop computers.

We list the instanton numbers in table 4. Observe that the instanton numbers sometimes do depend on the torsion part of their homology class.

5. The self-mirror property

When one speaks of a Calabi-Yau manifold Y being self-mirror, one has to indicate which

JHEP10(2007)023

level of invariants one is referring to. In particular, one might think of four types of invariants that are natural from the point of view of string theory. The weakest level is just the Euler number. In general, exchanging complex structure and Kähler moduli changes the sign of $\chi(Y) = 2h^{11}(Y) - 2h^{21}(Y)$. Therefore, a necessary condition for Y and its mirror Y^* to be equal is obviously that

$$\chi(Y) = -\chi(Y^*) = 0. \tag{5.1}$$

This level of invariants, however, is much too crude and therefore insufficient. A much stronger level is based on the fact that the cohomology groups of even degree come with an integral lattice structure and form a ring, and therefore have a product. Because of Poincaré duality, that is, $H^2(Y) = H^4(Y)^\vee$, it is sufficient to look at $H^2(Y)$. There is a product $H^2(Y)\times H^2(Y) \to H^2(Y)$ whose structure constants κ_{ijk} are the triple intersection numbers. These intersection numbers are finer invariants than just the dimensions of the cohomology groups, and a self-mirror Calabi-Yau threefold should satisfy

$$\kappa_{ijk}(Y) = \kappa_{ijk}(Y^*). \tag{5.2}$$

For simply connected threefolds with torsion-free homology a theorem of Wall [41] states that the cohomology groups with the intersection product $\kappa_{ijk}(Y)$ together with the second Chern class $\mathrm{c}_2(Y)$ determine the diffeomorphism type of Y.

If, however, Y and Y^* have non-trivial fundamental groups then we cannot conclude that easily that they are diffeomorphic. But the non-trivial fundamental group is often reflected in torsion in homology (for example if $\pi_1(Y)$ is Abelian). In that case, the conjecture of [6] says that for any Calabi-Yau threefold Z

$$H^3\big(Z,\mathbb{Z}\big)_{\text{tors}} \simeq H^2\big(Z^*,\mathbb{Z}\big)_{\text{tors}}, \qquad H^2\big(Z,\mathbb{Z}\big)_{\text{tors}} \simeq H^3\big(Z^*,\mathbb{Z}\big)_{\text{tors}}. \tag{5.3}$$

Therefore, a self-mirror manifold $Y = Y^*$ is expected to satisfy

$$H^2\big(Y,\mathbb{Z}\big)_{\text{tors}} \simeq H^3\big(Y,\mathbb{Z}\big)_{\text{tors}}. \tag{5.4}$$

Of the many spaces Y satisfying eq. (5.1) there are only a few which also satisfy eq. (5.2).

So far we only considered classical topology, but we know that the ring $H^2(Y)$ experiences quantum corrections when going far away from the large volume limit. At small volume the intersection numbers are replaced by the three-point functions $C_{ijk}(q)$ of (topological) conformal field theory in eq. (3.53). In the large volume limit q goes to zero and the $C_{ijk}(q)$ go to κ_{ijk}, as expected. The $C_{ijk}(q)$ are characterized by the genus zero instanton numbers $n_d^{(0)} = n_d$. In mathematical terms, these are resummations of the Gromow-Witten invariants of Y and characterize the symplectic structure of Y. This level of invariants is even stronger than the cohomology ring, since there are examples of diffeomorphic manifolds which have different Calabi-Yau structures, i.e. different $n_d^{(0)}$ [50, 51, 31]. Therefore, a self-mirror Calabi-Yau threefold Y must satisfy

$$n_d^{(0)}(Y) = n_d^{(0)}(Y^*). \tag{5.5}$$

One can go even further and couple the topological conformal field theory to topological gravity and define higher genus instanton numbers $n_d^{(g)}$, where now

$$n_d^{(g)}(Y) = n_d^{(g)}(Y^*), \qquad g > 0 \tag{5.6}$$

has to hold. These invariants are very difficult to compute, however see [52, 53] for recent progress. We do not know whether they contain more information about the symplectic structure than the genus zero invariants. In other words, there are presently no examples known whose $n_d^{(g)}$ agree for $g = 0$ but differ for $g > 0$.

Now, one can start with any Y and use some method to construct the mirror Y^*. Among these are the Greene-Plesser construction in conformal field theory, or its geometric generalizations by Batyrev and Borisov for complete intersections in toric varieties. Then, to show that Y is self-mirror one proceeds to compute the various invariants. The simplest condition, eq. (5.1), can directly be checked in terms of the toric data. This concretely means that one starts with a mirror pair Y and Y^* satisfying eq. (5.1) and checks whether eqs. (5.2), (5.4), (5.5), and (5.6) are satisfied. In fact, in section 4 we collected a large amount of evidence in favor of the claim that X and its Batyrev-Borisov mirror threefold X^* are the same. Indeed, eqs. (3.24), (4.11) and (4.44) show that $\widetilde{X}$, $\overline{X}$, and X satisfy by construction the constraint eq. (5.1) on the Euler number. More interestingly, by the identifications found in eqs. (4.37) and (4.43) we observed that the condition on the intersection ring, eq. (5.2), is satisfied for $\overline{X}$ and X, respectively. Next, eq. (4.25) and table 3 show that X also fulfils the requirement eq. (5.5) on the genus zero instanton numbers. It would be very interesting to see whether also the condition eq. (5.6) for higher genus curves can be met.

Finally, we consider the torsion in cohomology. In Part A section 5we have shown that

$$H^3\big(X, \mathbb{Z}\big)_{\text{tors}} \simeq H^2\big(X, \mathbb{Z}\big)_{\text{tors}} \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_3, \tag{5.7}$$

as we expect from a self-mirror threefold. Moreover, we can actually compute the fundamental group of the Batyrev-Borisov mirror independently. For that, first notice that the quotient $\overline{X}^* = \widetilde{X}^*/G_1^*$ is fixed-point free, see subsection 4.2. The mirror permutation G_2^* on $\overline{X}^*$ acts freely as well. Therefore, both X and X^* are free quotients by a group isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, thus their fundamental groups are

$$\pi_1\big(X\big) \simeq \pi_1\big(X^*\big) \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_3. \tag{5.8}$$

Moreover, on can easily show that on a proper[14] Calabi-Yau threefold Z one has $H^2(Z, \mathbb{Z})_{\text{tors}} = \pi_1(Z)_{\text{ab}}$, the Abelianization of the fundamental group. Hence, we see that

$$H^3(X, \mathbb{Z})_{\text{tors}} \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_3 \simeq H^2(X^*, \mathbb{Z})_{\text{tors}} \tag{5.9}$$

and the first of eq. (5.3) is true. This provides the first evidence for the conjecture of [6] in a context other than toric hypersurfaces.

[14] A proper Calabi-Yau threefold has holonomy group the full SU(3). In particular, this implies that the fundamental group is finite.

JHEP10(2007)023

Another point of view is that there is a geometrical or rather combinatorial reason for the self-mirror property in this case. From eqs. (3.20) and (3.23) one can easily see that the lattice points $\nu_i, \nu_{6+i}, \nu_{13}, \nu_{14}$, $i = 1, \ldots, 3$, span a sub-polytope of ∇^* satisfying the same linear relations as all the lattice points ρ_i of Δ^* in eq. (3.7). Hence, this sub-polytope is isomorphic to Δ^*. The same is true for the polytopes $\bar{\nabla}^*$ and $\bar{\Delta}^*$. The toric variety $\mathbb{P}_{\bar{\nabla}^*}$ which is the ambient space of $\overline{X}^*$ can therefore be regarded as a blow-up of a quotient of $\mathbb{P}_{\bar{\Delta}^*}$, the ambient space of $\overline{X}$. Actually, this blow-up makes all 7 divisors of $\overline{X}^*$ toric. Similarly, $\mathbb{P}_{\nabla^*}$ can be regarded as a blow-up of a quotient of $\mathbb{P}_{\Delta^*}$. As shown in subsection 3.3 this entails that all 19 Kähler moduli of $\widetilde{X}^*$ are realized torically. Note that it is possible that the mirror polytopes Δ^* and ∇^* are actually isomorphic. In fact, for toric hypersurfaces there are 41,710 self-dual polytopes [54]. The novel feature in our case is that non-isomorphic polytopes lead to self-mirror complete intersections, consistent with the nef partitions.

6. Factorization vs. the $(3, 1, 0, 0, 0)$ curve

One interesting observation is that the prepotential $\mathcal{F}^{\text{np}}_{X,0}$ at order p, see eq. (4.51) in this paper and eq. (8.34) in Part A [1], factors into $\sum_{i,j=0}^{2} b_1^i b_2^j$ times a function of p, q, r only. This means that the instanton number for any pseudo-section (curve contributing at order p) does not depend on the torsion part of its homology class. In other words, for any pseudo-section there are 8 other pseudo-sections with the same class in $H_2(X,\mathbb{Z})_{\text{free}}$ and together filling up all of $H_2(X,\mathbb{Z})_{\text{tors}} = \mathbb{Z}_3 \oplus \mathbb{Z}_3$. In contrast, this factorization does not hold at order p^3. For example,

$$\begin{aligned}\mathcal{F}^{\text{np}}_{X,0}(p,q,r,b_1,b_2) = \cdots &+ 3p^3 q \\ &+ 0\big(b_1 + b_1^2 + b_2 + b_1 b_2 + b_1^2 b_2 + b_2^2 + b_1 b_2^2 + b_1^2 b_2^2\big) p^3 q \\ &+ \cdots .\end{aligned} \tag{6.1}$$

The purpose of this subsection is to understand this behavior.

First, the factorization of the prepotential at any order of p not divisible by 3 follows from an extra symmetry that we have not utilized so far. The covering space $\widetilde{X}$ is, in addition to eqs. (2.3a) and (2.3b), also invariant under another $\hat{G} = \mathbb{Z}_3 \times \mathbb{Z}_3$ action generated by ($\zeta \overset{\text{def}}{=} e^{\frac{2\pi i}{3}}$)

$$\hat{g}_1 : \begin{cases} [x_0 : x_1 : x_2] \mapsto [x_0 : \zeta x_1 : \zeta^2 x_2] \\ [t_0 : t_1] \mapsto [t_0 : t_1] \text{ (no action)} \\ [y_0 : y_1 : y_2] \mapsto [y_0 : y_1 : y_2] \text{ (no action)} \end{cases} \tag{6.2a}$$

and

$$\hat{g}_2 : \begin{cases} [x_0 : x_1 : x_2] \mapsto [x_1 : x_2 : x_0] \\ [t_0 : t_1] \mapsto [t_0 : t_1] \text{ (no action)} \\ [y_0 : y_1 : y_2] \mapsto [y_0 : y_1 : y_2] \text{ (no action)} \end{cases} \tag{6.2b}$$

This symmetry has fixed points and, therefore, cannot be used if one is looking for a smooth quotient of $\widetilde{X}$. However, it commutes with G and hence descends to a $\hat{G} = \mathbb{Z}_3 \times \mathbb{Z}_3$

symmetry of X (with fixed points). Clearly, the instanton sum must observe this additional geometric symmetry. To make use of this symmetry, we have to express its action on the variables in $\mathcal{F}^{\mathrm{np}}_{X,0}(p,q,r,b_1,b_2)$. We can do so by first noting that the basic 81 curves

$$s_1 \underline{\times} s_2 \subset \widetilde{X}, \qquad s_1 \in MW(B_1),\ s_2 \in MW(B_2) \tag{6.3}$$

are really one orbit under $G \times \hat{G}$. Recall that, after dividing out G, these curves became the 9 sections in $MW(X) = \mathbb{Z}_3 \oplus \mathbb{Z}_3$, see Part A subsection 8.3. We now observe that $MW(X) = \{s_{ij}\}$ is one $\hat{G}$-orbit; since each of these sections contributes $pb_1^i b_2^j$, $i,j = 0,\dots,2$ the induced $\hat{G}$ action on the prepotential must be

$$\begin{aligned} \hat{g}_1 :\ & \mathcal{F}^{\mathrm{np}}_{X,0}(p,q,r,b_1,b_2) \mapsto \mathcal{F}^{\mathrm{np}}_{X,0}(b_1 p,q,r,b_1,b_2), \\ \hat{g}_2 :\ & \mathcal{F}^{\mathrm{np}}_{X,0}(p,q,r,b_1,b_2) \mapsto \mathcal{F}^{\mathrm{np}}_{X,0}(b_2 p,q,r,b_1,b_2). \end{aligned} \tag{6.4}$$

Clearly, the prepotential must be invariant under the $\hat{g}_1$, $\hat{g}_2$ action. While imposing no constraint on the p^{3n} terms in the prepotential, all other powers of p must appear in the combination

$$p^n \Big(\sum_{i,j=0}^{2} b_1^i b_2^j \Big), \qquad n \not\equiv 0 \mod 3. \tag{6.5}$$

This proves the factorization observed at the beginning of this subsection.

Second, we would like to understand the $p^3 q$ terms in eq. (6.1). These are the curves in the homology classes[15]

$$(3,1,0,*,*) \in \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 = H_2\big(X,\mathbb{Z}\big). \tag{6.6}$$

We will show that the rational curves in this class come in a single family, that is, the moduli space of genus 0 curves on X in these homology classes

$$\mathcal{M}_0\Big(X, (3,1,0,*,*)\Big) \tag{6.7}$$

is connected. In particular, all such curves have the same homology class $(3,1,0,0,0)$ and only contribute to $p^3 q$ in the prepotential eq. (6.1). As discussed in Part A section 5,any such map $C_X : \mathbb{P}^1 \to X$ factors

$$\begin{tikzcd} \mathbb{P}^1 \arrow[rr, "C_X"] \arrow[dr, "C_{\widetilde{X}}"'] & & X \\ & \widetilde{X} \arrow[ur, "q"'] & \end{tikzcd} . \tag{6.8}$$

The map $C_{\widetilde{X}}$ can be written in terms of homogeneous coordinates as a function

$$C_{\widetilde{X}} :\ \mathbb{P}^1_{[z_0:z_1]} \mapsto \mathbb{P}^2_{[x_0:x_1:x_2]} \times \mathbb{P}^1_{[t_0:t_1]} \times \mathbb{P}^2_{[y_0:y_1:y_2]} \tag{6.9}$$

[15] Recall that the exponent of p is the degree along the base $\mathbb{P}^1$. This is why we pick a basis in $H_2(X,\mathbb{Z})_{\mathrm{free}}$ such that a curve in (n_1,n_2,n_3,m_1,m_2) contributes at order $p^{n_1} q^{n_2} r^{n_3} b_1^{m_1} b_2^{m_2}$ in the prepotential.

satisfying the equations (2.2a) and (2.2b) defining $\widetilde{X}$,

$$F_1 \circ C_{\widetilde{X}}\big([z_0 : z_1]\big) = 0 = F_2 \circ C_{\widetilde{X}}\big([z_0 : z_1]\big) \qquad \forall [z_0 : z_1] \in \mathbb{P}^1 . \tag{6.10}$$

The curve C_X ends up in the homology class $(3, 1, 0, *, *)$ if and only if the defining equation (6.9) is of degree $(3, 1, 0)$ in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$. Hence, eq. (6.9) is defined by complex constants α_{ij}, β_{ij}, γ_i (up to rescaling) such that

$$\begin{aligned} x_i &= \alpha_{i0}\, z_0 + \alpha_{i1}\, z_1 & i &= 0,1,2 \\ t_i &= \beta_{i0}\, z_0^3 + \beta_{i1}\, z_0^2 z_1 + \beta_{i2}\, z_0 z_1^2 + \beta_{i3}\, z_1^3 & i &= 0,1 \\ y_i &= \gamma_i & i &= 0,1,2. \end{aligned} \tag{6.11}$$

These constants have to be picked such that the resulting curve lies on the complete intersection $\widetilde{X}$, that is, they have to satisfy eq. (6.10). Inserting eq. (6.11), we find that $F_1 \circ C_{\widetilde{X}}\big([z_0 : z_1]\big)$ is a homogeneous degree 6 polynomial in $[z_0 : z_1]$. Since the coefficients of $z_0^k z_1^{6-k}$ must vanish individually, this yields 7 constraints for the parameters α_{ij}, β_{ij}. What makes this system of constraint equations tractable is the fact that they are all linear in β_{ij},

$$F_1 \circ C_{\widetilde{X}} = 0 \quad \Leftrightarrow \quad \begin{pmatrix} A_1 & 0 & 0 & 0 & A_5 & 0 & 0 & 0 \\ A_2 & A_1 & 0 & 0 & A_6 & A_5 & 0 & 0 \\ A_3 & A_2 & A_1 & 0 & A_7 & A_6 & A_5 & 0 \\ A_4 & A_3 & A_2 & A_1 & A_8 & A_7 & A_6 & A_5 \\ 0 & A_4 & A_3 & A_2 & 0 & A_8 & A_7 & A_6 \\ 0 & 0 & A_4 & A_3 & 0 & 0 & A_8 & A_7 \\ 0 & 0 & 0 & A_4 & 0 & 0 & 0 & A_8 \end{pmatrix} \begin{pmatrix} \beta_{00} \\ \beta_{01} \\ \beta_{02} \\ \beta_{03} \\ \beta_{10} \\ \beta_{11} \\ \beta_{12} \\ \beta_{13} \end{pmatrix} = 0 \tag{6.12}$$

where

$$\begin{aligned} A_1 &\stackrel{\text{def}}{=} \alpha_{00}^3 + \alpha_{10}^3 + \alpha_{20}^3 & A_5 &\stackrel{\text{def}}{=} \alpha_{00}\alpha_{10}\alpha_{20} \\ A_2 &\stackrel{\text{def}}{=} 3\alpha_{01}\alpha_{00}^2 + 3\alpha_{11}\alpha_{10}^2 + 3\alpha_{21}\alpha_{20}^2 + \alpha_{20}^3 & A_6 &\stackrel{\text{def}}{=} (\alpha_{01}\alpha_{10} + \alpha_{00}\alpha_{11})\alpha_{20} + \alpha_{00}\alpha_{10}\alpha_{21} \\ A_3 &\stackrel{\text{def}}{=} 3\alpha_{01}^2\alpha_{00} + 3\alpha_{11}^2\alpha_{10} + 3\alpha_{21}^2\alpha_{20} & A_7 &\stackrel{\text{def}}{=} \alpha_{01}\alpha_{11}\alpha_{20} + (\alpha_{01}\alpha_{10} + \alpha_{00}\alpha_{11})\alpha_{21} \\ A_4 &\stackrel{\text{def}}{=} \alpha_{01}^3 + \alpha_{11}^3 + \alpha_{21}^3 & A_8 &\stackrel{\text{def}}{=} \alpha_{01}\alpha_{11}\alpha_{21}. \end{aligned} \tag{6.13}$$

Thinking of this as 7 linear equations for the 8 parameters β_{ij}, there is always a non-zero solution. The solution is generically unique up to an overall factor, and turns into an $\mathbb{P}^n$ for special values of the α_{ij}. Moreover, the parameter space of the α_{ij} is connected (essentially, the moduli space of lines in $\mathbb{P}^2$). Since we just identified the parameter space of the $(\alpha_{ij}, \beta_{ij})$ as a blow-up thereof, it is therefore connected as well.

It remains to satisfy $F_2 \circ C_{\widetilde{X}} = 0$. One can easily see that the only way is to pick the γ_i to be simultaneous solutions of

$$\gamma_0^3 + \gamma_1^3 + \gamma_2^3 = 0 = \gamma_1\gamma_2\gamma_3. \tag{6.14}$$

Since two cubics intersect in 9 points, there are 9 such solutions, permuted by G. Therefore, the parameter space of $(\alpha_{ij}, \beta_{ij}, \gamma_i)$ has 9 connected components, permuted by the G-action.

The moduli space of curves C_X on X is the G-quotient of the moduli space of curves $C_{\widetilde{X}}$ on $\widetilde{X}$, and therefore has only a single connected component. By continuity, every curve C_X in this connected family has the same homology class, explaining the piece of the prepotential given in eq. (6.1).

7. Towards a closed formula

Putting all the information together we found out about the prepotential on X, one can try to divine a closed form for the prepotential. We guess that the order p^n terms have the closed form

$$\mathcal{F}^{\mathrm{np}}_{X,0}(p,q,r,b_1,b_2)\Big|_{p^n} = \frac{p^n}{8^{n-1}}\left(\sum_{i,j\in\mathbb{Z}_3} b_1^i b_2^j\right)\Big(P(q)^4P(r)^4\Big)^n M_{2n-2}(q,r) \tag{7.1}$$

if n is not a multiple of 3 and, slightly weaker, that

$$\mathcal{F}^{\mathrm{np}}_{X,0}(p,q,r,1,1)\Big|_{p^n} = \frac{9p^n}{8^{n-1}}\Big(P(q)^4P(r)^4\Big)^n M_{2n-2}(q,r) \tag{7.2}$$

if n is a multiple of 3. Here,

- $P(q)$ is the usual generating function of partitions eq. (1.4).
- The M_{2n-2} are polynomials in the Eisenstein series $E_2(q)$, $E_4(q)$, $E_6(q)$ and $E_2(r)$, $E_4(r)$, $E_6(r)$, starting with

$$\begin{aligned}
M_{-2}(q,r) &= 0\\
M_0(q,r) &= 1\\
M_2(q,r) &= E_2(q)E_2(r)\\
M_4(q,r) &= \frac{13}{108}E_4(q)E_4(r) + \frac{1}{4}\Big(E_4(q)E_2(r)^2 + E_2(q)^2E_4(r)\Big) + \frac{7}{4}E_2(q)^2E_2(r)^2\\
M_6(q,r) &= \frac{1}{27}E_6(q)E_6(r) + \frac{13}{54}\Big(E_6(q)E_4(r)E_2(r) + E_4(q)E_2(q)E_6(r)\Big)\\
&\quad + \frac{1}{6}\Big(E_6(q)E_2(r)^3 + E_2(q)^3E_6(r)\Big) + \frac{79}{108}E_4(q)E_2(q)E_4(r)E_2(r)\\
&\quad + \frac{5}{4}\Big(E_2(q)^3E_4(r)E_2(r) + E_4(q)E_2(q)E_2(r)^3\Big) + \frac{47}{12}E_2(q)^3E_2(r)^3\\
M_8(q,r) &= \frac{2}{3}E_6(q)E_2(q)E_6(r)E_2(r) + \frac{1309}{6750}E_4(q)E_4(q)E_4(r)E_4(r)\\
&\quad + \frac{25}{108}\Big(E_6(q)E_2(q)E_4(r)E_4(r) + E_4(q)E_4(q)E_6(r)E_2(r)\Big)\\
&\quad + \frac{85}{54}\Big(E_6(q)E_2(q)E_4(r)E_2(r)^2 + E_4(q)E_2(q)^2E_6(r)E_2(r)\Big)\\
&\quad + \frac{13}{12}\Big(E_6(q)E_2(q)E_2(r)^4 + E_2(q)^4E_6(r)E_2(r)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{137}{216}\Big(E_4(q)E_4(q)E_4(r)E_2(r)^2+E_4(q)E_2(q)^2E_4(r)E_4(r)\Big)\\
&+\frac{3}{8}\Big(E_4(q)E_4(q)E_2(r)^4+E_2(q)^4E_4(r)E_4(r)\Big)\\
&+\frac{34}{9}E_4(q)E_2(q)^2E_4(r)E_2(r)^2+\frac{121}{12}E_2(q)^4E_2(r)^4\\
&+\frac{41}{8}\Big(E_4(q)E_2(q)^2E_2(r)^4+E_2(q)^4E_4(r)E_2(r)^2\Big).
\end{aligned}
\tag{7.3}
$$

They are symmetric under the exchange $q \leftrightarrow r$ and of weight $2n$ in q and r separately. But, for example, M_4 above does not factor into a function of q and a function of r. So the M_{2n-2} are not the products of the polynomials appearing in the dP_9 prepotential. However, by setting $q = 0$ or $r = 0$ one recovers the corresponding polynomials in the dP_9 prepotential [55].

- The E_{2i} are the usual Eisenstein series

$$
\begin{aligned}
E_2(q) &= 1 - 24q - 72q^2 - 96q^3 - 168q^4 - 144q^5 - 288q^6 + O(q^7)\\
E_4(q) &= 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 + O(q^6)\\
E_6(q) &= 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + O(q^5).
\end{aligned}
\tag{7.4}
$$

Note that the naive Taylor series coefficients of the prepotential are fractional, but when expanding in terms of Li_3's (which account for the multicover contributions) one finds integral instanton numbers.

These expressions for the prepotential agree with all instanton numbers computed in this paper. Unfortunately, we have not been able to guess a closed formula that includes the b_1 and b_2 dependence of the prepotential $\mathcal{F}^{\mathrm{np}}_{X,0}(p,q,r,b_1,b_2)|_{p^n}$ if n is divisible by 3. We expect that these involve extra functions beyond the Eisenstein series.

8. Conclusion

In the initial paper Part A [1], we analyzed the topology of the Calabi-Yau manifold of interest and found that

$$
H_2(X,\mathbb{Z}) = \mathbb{Z}^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3. \tag{8.1}
$$

Although the presence of torsion curve classes complicates the counting of rational curves, we managed to derive the A-model prepotential to linear order in p.

The goal of this paper is to go beyond the results of Part A using mirror symmetry. By carefully adapting methods designed for complete intersections in toric varieties, we can apply mirror symmetry to compute the instanton numbers on X, even though X is not toric. Using that X is self-mirror, we completely solve this problem and are able to calculate the complete A-model prepotential to any desired precision (and for arbitrary degrees in p), limited only by computer power. Carrying out this computation, we find the first examples of instanton numbers that do depend on the torsion part of their integral homology class, see table 4 on page 32.

Since the self-mirror property of X is important, we investigate it in detail. In doing so, we go far beyond just checking that the Hodge numbers are self-mirror. In particular,

JHEP10(2007)023

we find that the intersection rings are identical and that torsion in homology obeys the conjectured mirror relation [6]. Finally, going beyond classical geometry, we independently calculate certain instanton numbers on X and its Batyrev-Borisov mirror X^*. Again, we find that X and X^* are indistinguishable, providing strong evidence for X being self-mirror. Both of these results extend those found in Part A [1].

Using these results, we are able to guess certain closed expressions for the prepotential of X in terms of modular forms. In certain limits it specializes to the dP_9 prepotential of [55]. There it is known that the coefficients in p of the dP_9 prepotential satisfy a recursion relation. Moreover, there is a gap condition, that is, a certain number of subsequent terms in a series expansion is absent. This condition provides sufficient data to determine the integration constants for the recursion and allows to determine the prepotential completely, even at higher genus. We expect a similar story to be valid for the prepotential of X.

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A. Triangulation of $\bar{\nabla}^*$ and ∇^*

In principle the coherent triangulations of the fan over $\bar{\nabla}^*$ can be computed with TOPCOM by finding the 720 star triangulations in the total of $230,832$ coherent triangulations of $\bar{\nabla}^*$. The discussion of the symmetry properties is greatly facilitated, however, by an explicit understanding of their structure. We will work out the triangulations by first triangulating the facets and then checking the compatibility of their maximal intersections and the coherence of the resulting star triangulations.

We start with a couple of useful definitions. A circuit is a minimal collection of n affinely dependent points $p_1, \ldots, p_n$,

$$\lambda_1 p_1 + \ldots \lambda_n p_n = 0 \qquad \text{with} \qquad \lambda_1 + \ldots + \lambda_n = 0,\ \lambda_i \neq 0, \tag{A.1}$$

any proper subset of which is affinely independent. The coefficient vector λ_n hence has nonzero entries and is unique up to a prefactor. We indicate the unique separation into points with positive and negative coefficients with the notation $\langle p_{i_1} \ldots p_{i_s} | p_{i_{s+1}} \ldots p_{i_n} \rangle$.

JHEP10(2007)023

Each circuit admits two different triangulations, which are obtained by dropping one of the points with positive coefficients and one of the remaining points, respectively. We indicate this with a hat over the relevant subset. The two resulting triangulations

$$\langle \widehat{p_{i_1} \dots p_{i_s}} | p_{i_{s+1}} \dots p_{i_n} \rangle, \qquad \langle p_{i_1} \dots p_{i_s} | \widehat{p_{i_{s+1}} \dots p_{i_n}} \rangle \tag{A.2}$$

hence consist of s and $n-s$ simplices, respectively. If the first point is in the convex hull of the others, that is, $s=1$, then only one of the triangulations is maximal (all points are vertices of at least one simplex).

Furthermore, we introduce the notation:

$$\begin{aligned} a_i &= \bar{\nu}_i, & b_i &= \bar{\nu}_{3+i}, & c_i &= \bar{\nu}_{6+i}, & d_i &= \bar{\nu}_{12+i}, & i &= 1,2,3, \\ e &= \bar{\nu}_{13}, & f &= \bar{\nu}_{14}. \end{aligned} \tag{A.3}$$

Among these 14 vectors in eq. (A.3) there are 9 independent linear relations, see eq. (3.21),

$$\begin{gathered} a_1 + a_2 + a_3 = 0, \qquad c_1 + c_2 + c_3 = 0, \\ e + f = 0, \qquad b_i = a_i + e, \qquad d_l = c_l + f, \end{gathered} \tag{A.4}$$

which imply others like $a_i + b_j = a_j + b_i$ and $a_i + c_l = b_i + d_l$ or $e = \frac{1}{3}(b_1 + b_2 + b_3)$ and $f = \frac{1}{3}(d_1 + d_2 + d_3)$.

Lemma 1. *$\bar{\nabla}^*$ has 15 facets, 6 of which are simplicial:*

$$[a_i a_j b_i b_j c_l c_m d_l d_m]_{\substack{i<j, \\ l<m}} \quad [a_i a_j d_1 d_2 d_3]_{i<j}, \quad [b_1 b_2 b_3 c_l c_m]_{l<m}. \tag{A.5}$$

The nine non-simplicial facets form an orbit under the permutation symmetries $\mathbb{Z}_3^{ab} \times \mathbb{Z}_3^{cd}$ generated by $g_{ab} : \binom{a_i}{b_i} \to \binom{a_{i+1}}{b_{i+1}}$ and $g_{cd} : \binom{c_l}{d_l} \to \binom{c_{l+1}}{d_{l+1}}$. According to the linear relations eq. (A.4) the eight points on each non-simplicial facet form quadratic circuits $a_i + b_j = b_i + a_j$, $a_i + c_l = b_i + d_l$, and $c_l + d_m = c_m + d_l$, which we call *mixed* if they contain vertices of both elements of the nef partition $\langle a_i c_l | b_i d_l \rangle$, and *pure circuits* $\langle a_i b_j | b_i a_j \rangle$, $\langle c_l d_m | c_m d_l \rangle$ otherwise.

The coherent triangulations of the facets $[a_i a_j b_i b_j c_l c_m d_l d_m]$ are most easily obtained from their Gale transform

$$\begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}, \tag{A.6}$$

which is the coefficient matrix of the basis $a_i - a_j - b_i + b_j = 0$, $a_i - b_i + c_l - d_l = 0$ and $c_l - c_m - d_l + d_m = 0$ of linear relations. The coherent triangulations are in one-to-one correspondence to chambers that are seperated by the facets of the cones generated by all linear bases $\mu = \{v_1, v_2, v_3\}$ with v_i selected among the 8 column vectors of the Gale transform [57, 58]. In the present case the cones over the faces of the parallelepiped in figure 1 are subdivided into 24 chambers, which are indicated by dashed lines. The triangulations, which we can label by the facet containing and the edge adjoining

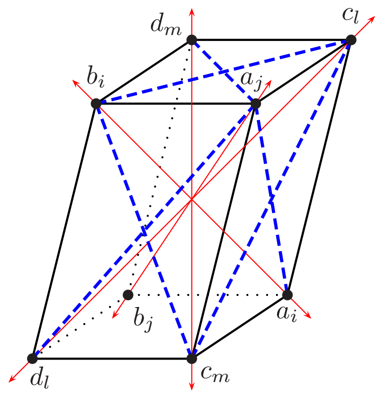
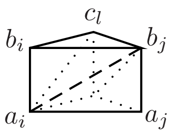


Figure 1: Secondary fan of the non-simplicial facets. Chambers are indicated by dashed lines.

the chamber, are obtained as the sets of complements of those bases μ that span a cone containing the respective chamber.

Hence, each non-simplicial facet has 24 coherent triangulations, which can be characterized by the triangulations of its 2 pure and of its 4 mixed circuits: Calling the triangulation $\langle\widehat{a_ic_l}|b_id_l\rangle$ positive and the triangulation $\langle a_ic_l|\widehat{b_id_l}\rangle$ negative, and arranging the cyclic permutations g_{ab} and g_{cd} in the horizontal and vertical direction, respectively, we can assign one of 16 different types $\boxed{\begin{smallmatrix} \pm & \pm \\ \pm & \pm \end{smallmatrix}}$ to each triangulation, where the signs indicate the induced triangulations of the mixed circuits. The constraints that reduce the a priori $32 = 2^6$ combinations to 24 all derive from the following rules:



$$\begin{array}{ccccc} \langle a_ic_l|\widehat{b_id_l}\rangle & \wedge & \langle\widehat{a_jc_l}|b_jd_l\rangle & \Rightarrow & \langle a_ib_j|\widehat{a_jb_i}\rangle \\ \langle\widehat{a_ic_l}|b_id_l\rangle & \wedge & \langle a_jc_l|\widehat{b_jd_l}\rangle & \Rightarrow & \langle\widehat{a_ib_j}|a_jb_i\rangle \end{array} \tag{A.7}$$

i.e. a triangular prism can be triangulated in 6 different ways, which correlates the a priori 8 combinations of the triangulations of the 3 squares (with analogous constraints for the two "horizontal" prisms $[a_ib_ic_lc_md_ld_m]$ contained in the facet $[a_ia_jb_ib_jc_lc_md_ld_m]$). Putting the pieces together we obtain

Lemma 2. *The* 24 *triangulations of the non-simplicial facets can be assorted as follows:*

- *For* $\boxed{\begin{smallmatrix} + & + \\ + & + \end{smallmatrix}}$, $\boxed{\begin{smallmatrix} - & - \\ - & - \end{smallmatrix}}$ *the pure circuits are unconstrained, yielding* $2 \cdot 2^2 = 8$ *triangulations.*

- *For* $\boxed{\begin{smallmatrix} + & + \\ - & - \end{smallmatrix}}$, $\boxed{\begin{smallmatrix} - & - \\ + & + \end{smallmatrix}}$ *the pure ab-circuit is unconstrained; with the transposed types* $\boxed{\begin{smallmatrix} + & - \\ + & - \end{smallmatrix}}$, $\boxed{\begin{smallmatrix} - & + \\ - & + \end{smallmatrix}}$ *this accounts for another* 8 *triangulations.*

- *The final* 8 *triangulations come from the* 8 *types with an odd number of positive signs, for which the triangulation of the pure circuits is unique.*

- *The two types* $\boxed{\begin{smallmatrix} + & - \\ - & + \end{smallmatrix}}$ *and* $\boxed{\begin{smallmatrix} - & + \\ + & - \end{smallmatrix}}$ *cannot occur because of contradictory implications for the triangulations of the pure circuits.*

The secondary fan and the induced triangulations for the codimension-two faces at which the non-simplical facets intersect can be obtained from figure 1 by projection along the dropped vertices. The secondary fan of the prism of eq. (A.7), for example, which is

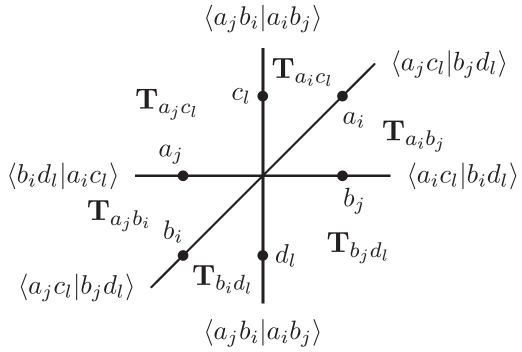


Figure 2: Secondary fan of the codimension two face $[a_i a_j b_i b_j c_l d_l]$.

+++ +++ +++	−++ +++ +++	−−+ +++ +++	−−− +++ +++	−−+ −++ +++	−−− −++ +++	−−+ −−+ +++
−−− −−− −−−	+−− −−− −−−	++− −−− −−−	+++ −−− −−−	++− +−− −−−	+++ +−− −−−	++− ++− −−−
$1 \cdot 2^6$	$9 \cdot 2^2$	$18 \cdot 2^2$	$6 \cdot 2^4$	$36 \cdot 2^0$	$36 \cdot 2^1$	$9 \cdot 2^2$

Table 5: The $824 = 2\,(64+36+72+96+36+72+36)$ star triangulations of ∇^*, including the $720 = 2\,(36+36+72+72+36+72+36)$ coherent triangulations.

shown in figure 2, is obtained from figure 1 by projection along the diagonal $\langle c_m d_m \rangle$. The wall crossings between the six cones in figure 2 are labeled by the circuits whose flops relate the adjoining triangulations [57].

For the construction of the complete star triangulation we now observe that the non-simplicial intersections of the 9 non-simplicial facets $[a_i a_j b_i b_j c_l c_m d_l d_m]$ are given by the 18 triangular prisms $[a_i a_j b_i b_j c_l d_l]$ and $[a_i b_i c_l c_m d_l d_m]$. If we interpret the former as vertices and the latter as links then the resulting compatibility conditions correspond to a graph with the topology of a torus. The vertices of this graph are decorated by signs as shown in table 5 and connected by horizontal and vertical links.

The restriction on the compatible signs is due to the absence of the inconsistent types $\boxed{\begin{smallmatrix}+&-\\-&+\end{smallmatrix}}$ and $\boxed{\begin{smallmatrix}-&+\\+&-\end{smallmatrix}}$ as subgraphs on the torus. The multiplicities $\mu \cdot 2^n$ come from the number n of unconstrained pure circuits and from the order μ of the effective part of the symmetry group generated by transposition and permutations of lines and columns. We thus find a total of 824 triangulations. The cyclic permutation symmetry that we want to keep on the Calabi-Yau manifold $\overline{X}^*$ amounts to a diagonal shift, i.e. its induced action on the graph is generated by $g_{ab} g_{cd}$. We are hence left with the types $\begin{smallmatrix}+&+&+\\+&+&+\\+&+&+\end{smallmatrix}$ and $\begin{smallmatrix}-&-&-\\-&-&-\\-&-&-\end{smallmatrix}$, and the shift symmetry furthermore aligns the triangulations of the pure circuits and thus reduced the multiplicities from 2^6 to 2^2, yielding a total of 8 triangulations for which $\mathbb{P}_{\overline{\nabla}^*}$ is G_2^* symmetric.

The resulting triangulations of the facet $[a_2 a_3 b_2 b_3 c_2 c_3 d_2 d_3]$ are

JHEP10(2007)023

$\substack{+++\\+++\\+++}$	triangulation of $[a_2a_3b_2b_3c_2c_3d_2d_3]$	
$\langle\widehat{a_2b_3}\vert a_3b_2\rangle$, $\langle\widehat{c_2d_3}\vert c_3d_2\rangle$	$\{[a_3b_2b_3d_2d_3],[b_2b_3c_3d_2d_3],[a_2a_3b_2d_2d_3],[b_2b_3c_2c_3d_2]\}$	
$\langle\widehat{a_2b_3}\vert a_3b_2\rangle$, $\langle c_2d_3\vert\widehat{c_3d_2}\rangle$	$\{[a_3b_2b_3d_2d_3],[b_2b_3c_2d_2d_3],[a_2a_3b_2d_2d_3],[b_2b_3c_2c_3d_3]\}$	(A.8)
$\langle a_2b_3\vert\widehat{a_3b_2}\rangle$, $\langle\widehat{c_2d_3}\vert c_3d_2\rangle$	$\{[a_2b_2b_3d_2d_3],[b_2b_3c_3d_2d_3],[a_2a_3b_3d_2d_3],[b_2b_3c_2c_3d_2]\}$	
$\langle a_2b_3\vert\widehat{a_3b_2}\rangle$, $\langle c_2d_3\vert\widehat{c_3d_2}\rangle$	$\{[a_2b_2b_3d_2d_3],[b_2b_3c_2d_2d_3],[a_2a_3b_3d_2d_3],[b_2b_3c_2c_3d_3]\}$	

$\substack{---\\---\\---}$	triangulation of $[a_2a_3b_2b_3c_2c_3d_2d_3]$	
$\langle a_2b_3\vert\widehat{a_3b_2}\rangle$, $\langle c_2d_3\vert\widehat{c_3d_2}\rangle$	$\{[a_2a_3b_3c_2c_3],[a_2a_3c_2c_3d_3],[a_2b_2b_3c_2c_3],[a_2a_3c_2d_2d_3]\}$	
$\langle a_2b_3\vert\widehat{a_3b_2}\rangle$, $\langle\widehat{c_2d_3}\vert c_3d_2\rangle$	$\{[a_2a_3b_3c_2c_3],[a_2a_3c_2c_3d_2],[a_2b_3b_2c_2c_3],[a_2a_3c_3d_2d_3]\}$	(A.9)
$\langle\widehat{a_2b_3}\vert a_3b_2\rangle$, $\langle c_2d_3\vert\widehat{c_3d_2}\rangle$	$\{[a_2a_3b_2c_2c_3],[a_2a_3c_2c_3d_3],[a_3b_2b_3c_2c_3],[a_2a_3c_2d_2d_3]\}$	
$\langle\widehat{a_2b_3}\vert a_3b_2\rangle$, $\langle\widehat{c_2d_3}\vert c_3d_2\rangle$	$\{[a_2a_3b_2c_2c_3],[a_2a_3c_2c_3d_2],[a_3b_2b_3c_2c_3],[a_2a_3c_3d_2d_3]\}$	

It can be checked that the triangulations listed in eqs. (A.8) and (A.9) come from the chambers contained in the cones over $[a_2a_3c_2c_3]$ and $[b_2b_3d_2d_3]$, respectively. For the first of these triangulations we consider the chamber adjoining the edge $[a_2c_2]$, which is contained in the span of the four bases $\mu_1 = \{a_2c_2c_3\}$, $\mu_2 = \{a_2a_3c_2\}$, $\mu_3 = \{b_3c_2c_3\}$ and $\mu_4 = \{a_2a_3d_3\}$, whose complements are $[a_3b_2b_3d_2d_3]$, $[b_2b_3c_3d_2d_3]$, $[a_2a_3b_2d_2d_3]$ and $[b_2b_3c_2c_3d_2]$ in agreement with the first triangulation in eq. (A.8).

Unfortunately, coherent triangulations of the facets that induce the same triangulations on their common (maximal) intersections do not automatically combine to *coherent* star triangulations of the polytope, and indeed only 720 of the 824 triangulations in table 5 turn out to be coherent. The non-coherent ones are easily isolated by observing that coherent triangulations (via their height functions) induce coherent triangulations of the prisms $[a_1a_2a_3b_1b_2b_2]$ and $[c_1c_2c_3d_1d_2d_3]$, which eliminates the triangulations for which $\mathbb{Z}_3^{ab}$ or $\mathbb{Z}_3^{cd}$ is not broken by the triangulation of the pure circuits. For the triangulation types $\substack{+++\\+++\\+++}$ and $\substack{---\\---\\---}$ this reduces the multiplicity from 8^2 to 6^2. The only other affected types are the ones in the middle column of table 5, which have unbroken horizontal symmetry and for which the multiplicity is reduced from $12\cdot 8$ to $12\cdot 6$. This poses a problem for the eight $\mathbb{Z}_3$-symmetric triangulations, which are all non-coherent. Coherence of the remaining 720 triangulations can be established by checking that their Mori cones are all strictly convex [59].

What comes to our rescue is that, even if all projective ambient spaces break the diagonal $\mathbb{Z}_3$ permutation symmetry, it may be preserved on $\overline{X}^*$ if the obstructing exceptional sets do not overlap with the complete intersection. In the present case these are the blow-ups of the singularities coming from the pure circuits, i.e. codimension two sets of the form $a_i \cdot b_j$ or $c_l \cdot d_m$, where we use, for simplicity, the symbol of the vertex $\bar{\nu}_j$ for the corresponding divisor D_j. Recall from eq. (4.5) that $\overline{X}^*$ is given by the product $\bar{D}^*_{0,1}\cdot\bar{D}^*_{0,2}$ of the divisors

$$\bar{D}^*_{0,1} = a_1 + a_2 + a_3 + b_1 + b_2 + b_3 + e, \qquad \bar{D}^*_{0,2} = c_1 + c_2 + c_3 + c_1 + d_2 + d_3 + f \quad \text{(A.10)}$$

defined by the nef partition. Taking into account the five linear equivalences, we observe that

$$\begin{aligned} a_1+b_1=a_2+b_2=a_3+b_3, \qquad\qquad & c_1+d_1=c_2+d_2=c_3+d_3, \\ b_1+b_2+b_3+e=d_1+d_2+d_3+f, \qquad\qquad & \end{aligned} \tag{A.11}$$

for divisor classes in the intersection ring. We first show that e and f do not intersect $\overline{X}^*$: In any maximal triangulation e and f belong only to the simplices

$$\big[\widehat{b_1b_2b_3}c_m c_l e\big], \qquad \big[\widehat{d_1d_2d_3}a_m a_l f\big], \tag{A.12}$$

respectively, so that

$$e\cdot a_i = e\cdot d_l = e\cdot f = 0 \quad \Rightarrow \quad e\cdot \bar{D}^*_{0,1} = e\cdot(b_1+b_2+b_3+e) = e\cdot(d_1+d_2+d_3+f) = 0 \tag{A.13}$$

and similarly $f\cdot \bar{D}^*_{0,2}=0$. Putting everything together, we conclude that

$$a_1\cdot b_2\cdot \bar{D}^*_{0,1} = a_1\cdot b_2\cdot 3(a_3+b_3) = 0 \tag{A.14}$$

because none of the facets, and hence no triangle in any of the triangulations contains $\{a_1,b_2,a_3\}$ or $\{a_1,b_2,b_3\}$ as a subset. Similarly $c_l\cdot d_m\cdot \bar{D}^*_{0,2}=0$ in the intersection ring for $l\neq m$. Consequently, all exceptional sets arising from triangulations of pure circuits do not intersect $\overline{X}^*$ and hence do not obstruct the cyclic permutation symmetry G_2^*. We will denote any of the remaining 36 coherent triangulations of type $\begin{smallmatrix}+&+&+\\+&+&+\\+&+&+\end{smallmatrix}$ and $\begin{smallmatrix}-&-&-\\-&-&-\\-&-&-\end{smallmatrix}$ by $T_+=T_+(\bar\nabla^*)$ and $T_-=T_-(\bar\nabla^*)$, respectively.

The polytope ∇^* of the mirror $\widetilde{X}^*$ of the universal cover has 39 lattice points, with the same 12 vertices as $\bar\nabla^*$ but living on the finer lattice $\bar{M}$. The 24 additional lattice points, see eq. (3.23), are

$$a_{ij} = \frac{1}{3}(a_i+2a_j), \qquad\qquad b_{ij} = \frac{1}{3}(b_i+2b_j), \tag{A.15}$$

$$c_{ij} = \frac{1}{3}(c_i+2c_j), \qquad\qquad d_{ij} = \frac{1}{3}(d_i+2d_j), \tag{A.16}$$

where $i\neq j$. These additional points are all located on edges of ∇^*. It is natural to consider triangulations that are refinements of the ones that we just discussed. Observing that the additional points turn all simplices in eqs. (A.8), (A.9) and (A.12) into pyramids over a tetrahedron with interior points on opposite edges it is easy to see that the maximal triangulations are unique and multiply the number $54=9\cdot 4+6\cdot 3$ of triangles in the original triangulations by a factor of 9. The resulting triangulations have been used to show that the divisors corresponding to the vertices a_{ij} and c_{ij} do not intersect $\tilde{X}^*$.

B. The flop of $\mathbf{X}^*$

In section 4.5 we have taken into account only one of the triangulations $T_+(\bar\nabla^*)$. We can repeat the same calculation with one of the triangulations T_-. We denote the resulting

Calabi-Yau manifold by $\overline{X}^*_-$. Skipping the details, we find that the generators of the Mori cone $\mathrm{NE}(\overline{X}^*_-)$ can be expressed in terms of those of $\mathrm{NE}(\overline{X}^*)$ in eq. (4.32) as

$$\begin{aligned}
\bar{l}^{(1)}_- &= \bar{l}^{(1)}_+ + 3\big(\bar{l}^{(4)}_+ + \bar{l}^{(5)}_+ + \bar{l}^{(6)}_+ + \bar{l}^{(7)}_+\big),\\
\bar{l}^{(2)}_- &= \bar{l}^{(2)}_+ + 3\big(\bar{l}^{(3)}_+ + \bar{l}^{(4)}_+ + \bar{l}^{(5)}_+\big),\\
\bar{l}^{(3)}_- &= \bar{l}^{(3)}_+ + \bar{l}^{(4)}_+,\\
\bar{l}^{(4)}_- &= -\,\bar{l}^{(4)}_+,\\
\bar{l}^{(5)}_- &= -\,\bar{l}^{(5)}_+ - \bar{l}^{(3)}_+ - \bar{l}^{(4)}_+ - \bar{l}^{(6)}_+ - \bar{l}^{(7)}_+,\\
\bar{l}^{(6)}_- &= \bar{l}^{(6)}_+,\\
\bar{l}^{(7)}_- &= \bar{l}^{(7)}_+.
\end{aligned} \tag{B.1}$$

One can also express the dual basis of divisors $\bar{J}'_i$ on the flop in terms of the dual basis $\bar{J}^*_i$ on X, see eq. (4.33). We find

$$\begin{array}{lll}
\bar{J}'_1 = \bar{J}^*_1, & \bar{J}'_2 = \bar{J}^*_2, & \bar{J}'_5 = 3\bar{J}^*_1 + 3\bar{J}^*_2 - \bar{J}^*_5\\
\bar{J}'_3 = 3\bar{J}^*_1 + \bar{J}^*_3 - \bar{J}^*_5, & \bar{J}'_4 = 3\bar{J}^*_1 + \bar{J}^*_3 - \bar{J}^*_4, & \\
\bar{J}'_6 = 3\bar{J}^*_2 - \bar{J}^*_5 + \bar{J}^*_6, & \bar{J}'_7 = 3\bar{J}^*_2 - \bar{J}^*_5 + \bar{J}^*_7. &
\end{array} \tag{B.2}$$

The intersection ring is

$$\begin{array}{rrrrr}
\bar{J}'^2_1\bar{J}'_2 = 1, & \bar{J}'^2_1\bar{J}'_3 = 2, & \bar{J}'^2_1\bar{J}'_4 = 1, & \bar{J}'^2_1\bar{J}'_5 = 3, & \bar{J}'^2_1\bar{J}'_6 = 3,\\
\bar{J}'^2_1\bar{J}'_7 = 3, & \bar{J}'_1\bar{J}'^2_2 = 1, & \bar{J}'_1\bar{J}'_2\bar{J}'_3 = 3, & \bar{J}'_1\bar{J}'_2\bar{J}'_4 = 3, & \bar{J}'_1\bar{J}'_2\bar{J}'_5 = 3,\\
\bar{J}'_1\bar{J}'_2\bar{J}'_6 = 3, & \bar{J}'_1\bar{J}'_2\bar{J}'_7 = 3, & \bar{J}'_1\bar{J}'^2_3 = 6, & \bar{J}'_1\bar{J}'_3\bar{J}'_4 = 6, & \bar{J}'_1\bar{J}'_3\bar{J}'_5 = 9,\\
\bar{J}'_1\bar{J}'_3\bar{J}'_6 = 9, & \bar{J}'_1\bar{J}'_3\bar{J}'_7 = 9, & \bar{J}'_1\bar{J}'^2_4 = 3, & \bar{J}'_1\bar{J}'_4\bar{J}'_5 = 9, & \bar{J}'_1\bar{J}'_4\bar{J}'_6 = 9,\\
\bar{J}'_1\bar{J}'_4\bar{J}'_7 = 9, & \bar{J}'_1\bar{J}'^2_5 = 9, & \bar{J}'_1\bar{J}'_5\bar{J}'_6 = 9, & \bar{J}'_1\bar{J}'_5\bar{J}'_7 = 9, & \bar{J}'_1\bar{J}'^2_6 = 9,\\
\bar{J}'_1\bar{J}'_6\bar{J}'_7 = 9, & \bar{J}'_1\bar{J}'^2_7 = 9, & \bar{J}'^2_2\bar{J}'_3 = 3, & \bar{J}'^2_2\bar{J}'_4 = 3, & \bar{J}'^2_2\bar{J}'_5 = 3,\\
\bar{J}'^2_2\bar{J}'_6 = 2, & \bar{J}'^2_2\bar{J}'_7 = 1, & \bar{J}'_2\bar{J}'^2_3 = 9, & \bar{J}'_2\bar{J}'_3\bar{J}'_4 = 9, & \bar{J}'_2\bar{J}'_3\bar{J}'_5 = 9,\\
\bar{J}'_2\bar{J}'_3\bar{J}'_6 = 9, & \bar{J}'_2\bar{J}'_3\bar{J}'_7 = 9, & \bar{J}'_2\bar{J}'^2_4 = 9, & \bar{J}'_2\bar{J}'_4\bar{J}'_5 = 9, & \bar{J}'_2\bar{J}'_4\bar{J}'_6 = 9,\\
\bar{J}'_2\bar{J}'_4\bar{J}'_7 = 9, & \bar{J}'_2\bar{J}'^2_5 = 9, & \bar{J}'_2\bar{J}'_5\bar{J}'_6 = 9, & \bar{J}'_2\bar{J}'_5\bar{J}'_7 = 9, & \bar{J}'_2\bar{J}'^2_6 = 6,\\
\bar{J}'_2\bar{J}'_6\bar{J}'_7 = 6, & \bar{J}'_2\bar{J}'^2_7 = 3, & \bar{J}'^3_3 = 18, & \bar{J}'^2_3\bar{J}'_4 = 18, & \bar{J}'^2_3\bar{J}'_5 = 27,\\
\bar{J}'^2_3\bar{J}'_6 = 27, & \bar{J}'^2_3\bar{J}'_7 = 27, & \bar{J}'_3\bar{J}'^2_4 = 18, & \bar{J}'_3\bar{J}'_4\bar{J}'_5 = 27, & \bar{J}'_3\bar{J}'_4\bar{J}'_6 = 27,\\
\bar{J}'_3\bar{J}'_4\bar{J}'_7 = 27, & \bar{J}'_3\bar{J}'^2_5 = 27, & \bar{J}'_3\bar{J}'_5\bar{J}'_6 = 27, & \bar{J}'_3\bar{J}'_5\bar{J}'_7 = 27, & \bar{J}'_3\bar{J}'^2_6 = 27,\\
\bar{J}'_3\bar{J}'_6\bar{J}'_7 = 27, & \bar{J}'_3\bar{J}'^2_7 = 27, & \bar{J}'^3_4 = 9, & \bar{J}'^2_4\bar{J}'_5 = 27, & \bar{J}'^2_4\bar{J}'_6 = 27,\\
\bar{J}'^2_4\bar{J}'_7 = 27, & \bar{J}'_4\bar{J}'^2_5 = 27, & \bar{J}'_4\bar{J}'_5\bar{J}'_6 = 27, & \bar{J}'_4\bar{J}'_5\bar{J}'_7 = 27, & \bar{J}'_4\bar{J}'^2_6 = 27,\\
\bar{J}'_4\bar{J}'_6\bar{J}'_7 = 27, & \bar{J}'_4\bar{J}'^2_7 = 27, & \bar{J}'^3_5 = 27, & \bar{J}'^2_5\bar{J}'_6 = 27, & \bar{J}'^2_5\bar{J}'_7 = 27,\\
\bar{J}_5\bar{J}'^2_6 = 27, & \bar{J}'_5\bar{J}'_6\bar{J}'_7 = 27, & \bar{J}_5\bar{J}'^2_7 = 27, & \bar{J}'^3_6 = 18, & \bar{J}'^2_6\bar{J}'_7 = 18,\\
\bar{J}'_6\bar{J}'^2_7 = 18, & \bar{J}'^3_7 = 9. & & &
\end{array} \tag{B.3}$$

JHEP10(2007)023

The second Chern class is

$$\begin{aligned}
&c_2\left(\overline{X}^*_-\right)\cdot \bar{J}'_1 = 12, && c_2\left(\overline{X}^*_-\right)\cdot \bar{J}'_5 = 18, && c_2\left(\overline{X}^*_-\right)\cdot \bar{J}'_2 = 12,\\
&c_2\left(\overline{X}^*_-\right)\cdot \bar{J}'_3 = c_2\left(\overline{X}^*_-\right)\cdot \bar{J}'_6 = 24, && c_2\left(\overline{X}^*_-\right)\cdot \bar{J}'_4 = c_2\left(\overline{X}^*_-\right)\cdot \bar{J}'_7 = 30. &&
\end{aligned} \tag{B.4}$$

We observe that both the intersection ring and the second Chern class cannot be brought into (2.6) and (4.38) by a linear transformation with integer coefficients, respectively. Hence, the second phase really is topologically distinct.

We denote the Fourier-transformed variables in the B-model prepotential (3.55) by q'_i, $i = 1, \ldots, 7$. With this notation, we obtain

$$\begin{aligned}
\mathcal{F}^B_{\overline{X}_-,0}(q'_1,\ldots,q'_7) = {}& 3\,q'_1 + 3\,q'_2 + 3\,q'_5 - \frac{45}{8}\,{q'_1}^2 - \frac{45}{8}\,{q'_2}^2 + \frac{3}{8}\,{q'_5}^2 + 3\,q'_3q'_5 + 3\,q'_5q'_6\\
&+ \frac{244}{9}\,{q'_1}^3 + \frac{244}{9}\,{q'_2}^3 + \frac{1}{9}\,{q'_5}^3 + 3\,q'_3q'_4q'_5 + 3\,q'_3q'_5q'_6 + 3\,q'_5q'_6q'_7\\
&- \frac{12333}{64}\,{q'_1}^4 - \frac{12333}{64}\,{q'_2}^4 + \frac{3}{64}\,{q'_5}^4 + 3\,q'_1{q'_4}^3 + 3\,q'_2{q'_7}^3\\
&+ \frac{3}{8}\,{q'_5}^2{q'_6}^2 + \frac{3}{8}\,{q'_3}^2{q'_5}^2 - 6\,q'_1q'_3q'_4q'_5 - 6\,q'_2q'_5q'_6q'_7\\
&+ 3\,q'_3q'_4q'_5q'_6 + 3\,q'_3q'_5q'_6q'_7 + O(q'^5).
\end{aligned} \tag{B.5}$$

The instanton numbers on $\overline{X}^*_-$ are the expansion coefficients in

$$\mathcal{F}^{\mathrm{np}}_{\overline{X}^*_-,0}(q'_1,\ldots,q'_7,1) = \sum_{n_1,\ldots,n_7} n^{\overline{X}^*_-}_{(n_1,n_2,n_3,n_4,n_5,n_6,n_7)}\,\mathrm{Li}_3\Big(\prod_{i=1}^{7} {q'_i}^{n_i}\Big). \tag{B.6}$$

Up to degree 4, they read

$$\begin{array}{llll}
n^{\overline{X}^*_-}_{1,0,0,0,0,0,0} = 3 & n^{\overline{X}^*_-}_{0,1,0,0,0,0,0} = 3 & n^{\overline{X}^*_-}_{0,0,0,0,1,0,0} = 3 & n^{\overline{X}^*_-}_{2,0,0,0,0,0,0} = -6\\
n^{\overline{X}^*_-}_{0,2,0,0,0,0,0} = -6 & n^{\overline{X}^*_-}_{3,0,0,0,0,0,0} = 27 & n^{\overline{X}^*_-}_{0,3,0,0,0,0,0} = 27 & n^{\overline{X}^*_-}_{4,0,0,0,0,0,0} = -192\\
n^{\overline{X}^*_-}_{0,4,0,0,0,0,0} = -192 & n^{\overline{X}^*_-}_{0,0,1,0,1,0,0} = 3 & n^{\overline{X}^*_-}_{0,0,0,0,1,1,0} = 3 & n^{\overline{X}^*_-}_{0,0,1,1,1,0,0} = 3\\
n^{\overline{X}^*_-}_{0,0,1,0,1,1,0} = 3 & n^{\overline{X}^*_-}_{0,0,0,0,1,1,1} = 3 & n^{\overline{X}^*_-}_{1,0,0,3,0,0,0} = 3 & n^{\overline{X}^*_-}_{0,1,0,0,0,0,3} = 3\\
n^{\overline{X}^*_-}_{1,0,1,1,1,0,0} = -6 & n^{\overline{X}^*_-}_{0,1,0,0,1,1,1} = -6 & n^{\overline{X}^*_-}_{0,0,1,1,1,1,0} = 3 & n^{\overline{X}^*_-}_{0,0,1,0,1,1,1} = 3
\end{array} \tag{B.7}$$

It is easy to check that the symmetry G^*_2 acts without fixed points on $\overline{X}^*_-$ so that there are two phases of the quotient X^*, too, with $h^{1,1}(X^*) = h^{1,2}(X^*) = 3$ and fundamental group $\pi_1(X^*) = \mathbb{Z}_3 \times \mathbb{Z}_3$. They correspond to the two classes of triangulations T_+ and T_-. The first phase was studied in detail in section 4.5 and section 4.6.

We denote the Calabi-Yau manifold in the second phase by $X^*_- = \overline{X}^*_-/G^*_2$. From the linear equivalence relations eq. (4.34) and the definition eq. (B.2) we can compute the induced group action on $H^2(\overline{X}^*_-, \mathbb{Z})$ and find

$$g_2^* \begin{pmatrix} \bar{J}'_1 \\ \bar{J}'_2 \\ \bar{J}'_3 \\ \bar{J}'_4 \\ \bar{J}'_5 \\ \bar{J}'_6 \\ \bar{J}'_7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 3 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 & -1 \\ 0 & 3 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \bar{J}'_1 \\ \bar{J}'_2 \\ \bar{J}'_3 \\ \bar{J}'_4 \\ \bar{J}'_5 \\ \bar{J}'_6 \\ \bar{J}'_7 \end{pmatrix} . \tag{B.8}$$

In terms of the three invariant divisors $J'_1 = \bar{J}'_5, J'_2 = \bar{J}'_1, J'_3 = \bar{J}'_2$ the intersection ring and the second Chern class of X^*_- then are

$$\begin{aligned} &J'^2_2 J'_3 = 1, \quad J'_1 J'^2_2 = 3, \quad J'_2 J'^2_3 = 1, \quad J'_1 J'_2 J'_3 = 3, \\ &J'^2_1 J'_2 = 9, \quad J'_1 J'^2_3 = 3, \quad J'^2_1 J'_3 = 9, \quad J'^3_1 = 27, \\ &\mathrm{c}_2\, J'_1 = 18, \quad \mathrm{c}_2\, J'_2 = 12, \quad \mathrm{c}_2\, J'_3 = 12. \end{aligned} \tag{B.9}$$

Again, we observe that there is no linear basis transformation with integer coefficients that brings both the intersection ring and the second Chern class into (4.18). Hence, also the phase X^*_- is topologically distinct from X^*.

To give a geometrical interpretation of what happens, we look at the induced action of G^*_2 on the toric Mori cone $\mathrm{NE}(\overline{X}^*_-)$. Only the generators $\bar{l}^{(1)}_-$, $\bar{l}^{(2)}_-$ and $\bar{l}^{(5)}_-$ in eq. (B.1) are invariant. This is exactly as in the first phase. Denoting the invariant generators $\bar{l}^{(5)}_\pm, \bar{l}^{(1)}_\pm, \bar{l}^{(2)}_\pm$ by $l^{(1)}_\pm, l^{(2)}_\pm, l^{(3)}_\pm$, respectively, we observe that phase T_- is obtained from the phase T_+ as a flop by the curve corresponding to the generator $l^{(1)}_+$:

$$l^{(1)}_- = -l^{(1)}_+, \qquad l^{(2)}_- = l^{(2)}_+ + 3l^{(1)}_+, \qquad l^{(3)}_- = l^{(3)}_+ + 3l^{(1)}_+. \tag{B.10}$$

If we use the realization of X in terms of the fiber product of two dP_9 surfaces, the above result means that the base $\mathbb{P}^1$ of X has been flopped.

Furthermore, having computed the G^*_2-action in eq. (B.8), we determine the descent equation for the prepotential to be

$$\mathcal{F}^{\mathrm{np}}_{X^*_-,0}\big(p', q', r', b'_1, b'_2\big) = \frac{1}{|G^*_2|} \mathcal{F}^{\mathrm{np}}_{\overline{X}^*_-,0}\big(p', q', b'_2, b'_2, r', {b'_2}^2, {b'_2}^2, b'_1\big). \tag{B.11}$$

The corresponding instanton numbers

$$\mathcal{F}^{\mathrm{np}}_{X^*_-,0}(p', q', r', 1, b'_2) = \sum_{n_1,n_2,n_3,m_2} n^{X^*_-}_{(n_1,n_2,n_3,m_2)}\, \mathrm{Li}_3\left(p'^{n_1} q'^{n_2} r'^{n_3} {b'_2}^{m_2}\right) \tag{B.12}$$

are listed in table 6.

(n_1, n_2, n_3)	$n^{X_-^*}_{(n_1,n_2,n_3,0)}$	$n^{X_-^*}_{(n_1,n_2,n_3,1)}$	$n^{X_-^*}_{(n_1,n_2,n_3,2)}$	$\sum_{m_2=0}^{2} n^{X_-^*}_{(n_1,n_2,n_3)}$
$(1,0,0)$	3	0	0	3
$(2,0,0)$	-6	0	0	-6
$(3,0,0)$	18	0	0	18
$(0,1,0)$	3	3	3	9
$(1,1,0)$	-6	-6	-6	-18
$(1,1,1)$	12	12	12	36
$(2,1,0)$	15	15	15	45
$(1,2,0)$	12	12	12	36

Table 6: Instanton numbers $n^{X_-^*}_{(n_1,n_2,n_3,m_2)}$ computed by toric mirror symmetry. They are invariant under the exchange $n_2 \leftrightarrow n_3$, so we only display them for $n_2 \leq n_3$.

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JHEP10(2007)023

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JHEP10(2007)023